

Particles and Fields

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Preface

Quantum field theory originally meant the theory of quantizing classical fields. Nowadays this notion is used for the general theory of quantum systems with **infinitely many degrees of freedom**. This is a vast subject, but we can cover only a few topics.

Warning: This manuscript is just a condensed set of notes for the lecturer. Obviously, the explanations given here are insufficient for the non-expert. Maybe they will be extended in the future. For the time being these notes should be considered no more than an outline of the stuff to be elaborated in the lectures.

Purely mathematical proofs will mainly be skipped and replaced by suitable references.

Recommended Literature: ([Araki, 1999](#); [Baumgärtel, 1995](#); [Bogolubov et al., 1990](#); [Borchers, 1996](#); [Buchholz, 1985](#); [Fredenhagen, 1995](#); [Haag, 1992](#); [Horuzhy, 1990](#); [Jost, 1965](#); [Streater and Wightman, 1989](#))

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Chapter 1

General Quantum Theory

1.1 Basic Logical Structure

1.1.1 Classical Logic and General Notions¹

Let \mathcal{L} be a set of propositions fulfilling

$$E_1, E_2 \in \mathcal{L} \implies E_1 \wedge E_2, \neg E_1 \in \mathcal{L}.$$

In common sense logic $E_1 \wedge E_2$ holds if and only if both E_1 and E_2 hold while $\neg E_1$ holds if and only if E_1 does not hold. Let us identify E_1 with E_2 whenever E_1 holds if and only if E_2 holds. Under these circumstances

$$E_1 \preceq E_2 \stackrel{\text{def}}{\iff} E_1 \wedge E_2 = E_1 \tag{1.1}$$

defines a *semi-ordering* \preceq on \mathcal{L} , i.e. the ordinary logical implication \preceq is

$$\begin{aligned} \textit{reflexive:} & & E & \preceq E, \\ \textit{transitive:} & & E_1 \preceq E_2, E_2 \preceq E_3 & \implies E_1 \preceq E_3, \\ \textit{anti-symmetric:} & & E_1 \preceq E_2, E_2 \preceq E_1 & \implies E_1 = E_2. \end{aligned}$$

The semi-ordered set (\mathcal{L}, \preceq) is even a *lattice*, i.e. for any two elements $E_1, E_2 \in \mathcal{L}$ there is an infimum as well as a supremum, namely

$$E_1 \wedge E_2 = \inf_{\mathcal{L}} \{E_1, E_2\}, \quad E_1 \vee E_2 = \sup_{\mathcal{L}} \{E_1, E_2\}, \tag{1.2}$$

where $E_1 \vee E_2$ holds iff at least one of the propositions E_1, E_2 holds.² This lattice has a *universal lower bound*

$$\check{0} \stackrel{\text{def}}{=} \inf_{\mathcal{L}} \mathcal{L}$$

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¹The definitions and notions introduced here are in agreement with those of (Birkhoff, 1967) and (Varadarajan, 2007).

²For lattices in general (1.2) serves as a definition for \wedge and \vee , consistent with (1.1).

as well as a *universal upper bound*

$$\check{1} \stackrel{\text{def}}{=} \sup_{\mathcal{L}} \mathcal{L}.$$

The negation \neg is an *orthocomplementation*, i.e. for all $E, E_1, E_2 \in \mathcal{L}$ we have

$$\begin{aligned} (O_1) : & E \wedge \neg E = \check{0}, \\ (O_2) : & E \vee \neg E = \check{1}, \\ (O_3) : & \neg(\neg E) = E, \\ (O_4) : & E_1 \preceq E_2 \implies \neg E_2 \preceq \neg E_1. \end{aligned}$$

The orthocomplemented lattice $(\mathcal{L}, \preceq, \neg)$ is *weakly modular*, i.e.:

$$E_1 \preceq E_2 \implies E_2 = (E_2 \wedge E_1) \vee (E_2 \wedge \neg E_1).$$

A *logic* is defined to be a weakly modular orthocomplemented lattice $(\mathcal{L}, \preceq, \neg)$ which is σ -complete; i.e. in which $\inf_{\mathcal{L}} A$ exists for every countable subset A of \mathcal{L} . A logic $(\mathcal{L}, \preceq, \neg)$ is called *classical* if – as in the above example – it is *distributive*; i.e.:³

$$\begin{aligned} (D_1) : & E_1 \wedge (E_2 \vee E_3) = (E_1 \wedge E_2) \vee (E_1 \wedge E_3), \\ (D_2) : & E_1 \vee (E_2 \wedge E_3) = (E_1 \vee E_2) \wedge (E_1 \vee E_3). \end{aligned}$$

An *ordered* pair $(E_1, E_2) \in \mathcal{L} \times \mathcal{L}$ is called *compatible* if⁴

$$E_1 = (E_1 \wedge E_2) \vee (E_1 \wedge \neg E_2). \quad (1.3)$$

A *probability measure* on a logic $(\mathcal{L}, \preceq, \neg)$ is defined to be a mapping w from \mathcal{L} into the interval $[0, 1]$ fulfilling the following two conditions:⁵

$$\begin{aligned} (W_1) : & w(1) = 1. \\ (W_2) : & w \text{ is } \sigma\text{-additive; i.e.:} \\ & w(\sup\{E_j : j = 1, 2, \dots\}) = \sum_j w(E_j) \text{ if } E_j \preceq \neg E_k \text{ for } j \neq k. \end{aligned}$$

1.1.2 Quantum Logic

Every known concrete quantum theory is a *statistical theory* of the following type. It is affiliated with

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³Thanks to orthocomplementation, (D_1) and (D_2) are equivalent.

⁴An orthocomplemented lattice $(\mathcal{L}, \preceq, \neg)$ is weakly modular if and only if

$$(E_1, E_2) \text{ compatible} \iff (E_2, E_1) \text{ compatible}$$

holds for all $E_1, E_2 \in \mathcal{L}$ (Birkhoff, 1967, Theorem 21, p. 53).

⁵Actually, one should also make sure that $w(E_1) = w(E_2) = 1 \implies w(E_1 \wedge E_2) = 1$ holds for all $E_1, E_2 \in \mathcal{L}$.

1. A set \mathcal{Q} of **macroscopic** prescriptions for preparing a ‘state’ of the system under consideration.
2. A set \mathcal{X} of **macroscopic** prescriptions for performing idealized *simple tests* (called *questions* by PIRON) on the system under consideration⁶ with only two possible outcomes referred to as ‘yes’ or ‘no’.
3. A mapping⁷

$$w : \mathcal{Q} \times \mathcal{X} \longrightarrow [0, 1]$$

with the following interpretation:⁸

$w(S, T)$ is the probability⁹ for the outcome ‘yes’ when performing a simple ‘test’ corresponding to T on the system in a ‘state’ corresponding to S .

Obviously, the ‘tests’ $T \in \mathcal{X}$ cannot discriminate elements $S_1, S_2 \in \mathcal{Q}$, which are equivalent in the following sense:

$$S_1 \sim S_2 \stackrel{\text{def}}{\iff} w(S_1, T) = w(S_2, T) \quad \forall T \in \mathcal{X}.$$

Similarly the ‘states’ $S \in \mathcal{Q}$ cannot discriminate ‘tests’ $T_1, T_2 \in \mathcal{X}$, which are equivalent in the sense that

$$T_1 \sim T_2 \stackrel{\text{def}}{\iff} w(S, T_1) = w(S, T_2) \quad \forall S \in \mathcal{Q}.$$

Therefore the appropriate mathematical formalism deals with the equivalence classes $[S]$ (also called *states*) and $[T]$ (also called *propositions* or *questions*) together with the (consistent) assignment

$$\omega(\hat{P}) \stackrel{\text{def}}{=} w(S, T) \quad \text{for } \omega = [S], \hat{P} = [T]$$

rather than the specific prescriptions S, T and the mapping w .

“What we call a state nowadays might turn out to be an equivalence class of states at later times. But this is only possible after having discovered new observables and new states at the same time because states and observables must be mutually separating.” (Borchers, 1996, p. 2)

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⁶In relativistic quantum field theory we cannot assume that these tests can be performed within arbitrarily small time intervals. Therefore, as becomes evident by Theorem 1.2.2, we work in the so-called HEISENBERG *picture*, in which time evolution is attributed to the ‘tests’ rather than to the ‘states’.

⁷Actually – as well known for open systems (Davies, 1976) – the probability for the outcome ‘yes’ or ‘no’ in a test performed before the ‘state’ is prepared need not have any meaning. However for all known models of **closed** quantum systems the ‘states’ can be imagined as having been prepared as early as one likes. This is essential for standard scattering theory.

⁸We do **not** claim that S uniquely characterizes a **microscopic** state, nor do we claim that T fixes the microscopic details of a test!

⁹Compare, e.g. (Peres, 1995, p. 25).

\mathcal{Q} and \mathcal{X} are always (more or less implicitly) chosen such that the following three conditions are fulfilled:¹⁰

(I₁): For every $\hat{P} \in \mathcal{L} \stackrel{\text{def}}{=} \{[T] : T \in \mathcal{X}\}$ there is also an element $\neg\hat{P} \in \mathcal{L}$ fulfilling¹¹

$$\omega(\neg\hat{P}) = 1 - \omega(\hat{P}) \quad \text{for all } \omega \in \mathcal{S} \stackrel{\text{def}}{=} \{[S] : S \in \mathcal{Q}\} .$$

(I₂): Let $\hat{P}_1, \hat{P}_2, \dots \in \mathcal{L}$. Then there is an element $\hat{I} \in \mathcal{L}$ such that for all $\omega \in \mathcal{S}$

$$\omega(\hat{I}) = 1 \text{ if and only if } \omega(\hat{P}_j) = 1 \text{ for } j = 1, 2, \dots$$

(I₃): Let $\hat{P}_1, \hat{P}_2, \dots \in \mathcal{L}$ be such that

$$\omega(\hat{P}_j) = 1 \implies \omega(\hat{P}_k) = 0 \quad \text{for all } \omega \in \mathcal{S} \quad \text{whenever } j < k .$$

Then there is an element $\hat{S} \in \mathcal{L}$ fulfilling

$$\omega(\hat{S}) = \omega(\hat{P}_1) + \omega(\hat{P}_2) + \dots \quad \forall \omega \in \mathcal{S} .$$

Note that (I₁) defines a mapping $\neg : \mathcal{L} \longrightarrow \mathcal{L}$. Moreover, there is always a natural semi-ordering of the elements of \mathcal{L} given by

$$\hat{P}_1 \preceq \hat{P}_2 \stackrel{\text{def}}{\iff} \omega(\hat{P}_1) \leq \omega(\hat{P}_2) \quad \forall \omega \in \mathcal{S} . \quad (1.4)$$

Theorem 1.1.1 (Structure Theorem) *If $\mathcal{L} \neq \emptyset$ and \mathcal{S} fulfill conditions (I₁)–(I₃), then $(\mathcal{L}, \preceq, \neg)$, with \preceq given by (1.4) and \neg given by (I₁), is a **logic**, i.e. a σ -complete weakly modular lattice (\mathcal{L}, \preceq) . Moreover, under these conditions, every $\omega \in \mathcal{S}$ is a probability measure over $(\mathcal{L}, \preceq, \neg)$ fulfilling the **JAUCH-PIRON condition***

$$\left(\omega(\hat{P}_1) = 1 = \omega(\hat{P}_2) \implies \omega(\hat{P}_1 \wedge \hat{P}_2) = 1 \right) \quad \forall \hat{P}_1, \hat{P}_2 \in \mathcal{L} . \quad (1.5)$$

Proof: See (Doebner and Lücke, 1991, appendix) (see also (Maczyński, 1974) for related results). ■

¹⁰These conditions are designed to allow for classical reasoning as far as possible. Implicit in (I₃) and (I₁) is the following *standardization postulate*: For every $\hat{P} \in \mathcal{L} \setminus \{\hat{0}\}$ there exist a state $\omega \in \mathcal{S}$ with $\omega(\hat{P}) = 1$. Therefore semi-transparent windows, e.g., cannot be used for simple tests.

¹¹A more general framework, allowing for nonlinear time evolution, was suggested in (Mielnik, 1974).

1.1.3 Quantum Reasoning

It seems natural to assign ‘actual’ properties $E_{\hat{P}}$ to the elements \hat{P} of \mathcal{L} in the sense that:

A system in the state $\omega \in \mathcal{S}$ has property $E_{\hat{P}}$ **with certainty** if and only if¹² $\omega(\hat{P}) = 1$. (1.6)

We are used giving names to these properties like ‘spin up’, ‘positive energy’ and so on. However, there is no evidence for the assumption that under all circumstances – independent of any test – the system has either property $E_{\hat{P}}$ or property $E_{\neg\hat{P}}$ – even though

$$\omega(\neg\hat{P}) = 1 - \omega(\hat{P}) \quad \forall \omega \in \mathcal{S}, \hat{P} \in \mathcal{L}$$

and even though tests corresponding to \hat{P} and $\neg\hat{P}$ can typically be performed jointly.¹³ This also becomes clear by the following lemma.

Lemma 1.1.2 (D. Pfeil) *For every finite set $\check{\mathcal{L}}$ there is a **classical** logic $(\mathcal{B}, \preceq_B, \neg_B)$ and a mapping $M : \check{\mathcal{L}} \rightarrow \mathcal{B}$ for which the following holds:*

For every mapping $\omega : \check{\mathcal{L}} \rightarrow [0, 1]$ there is a probability measure μ on $(\mathcal{B}, \preceq_B, \neg_B)$ fulfilling

$$\omega(\hat{P}) = \mu(M(\hat{P})) \quad \forall \hat{P} \in \check{\mathcal{L}}$$

and

$$\hat{P}_1 \neq \hat{P}_2 \implies \mu(M(\hat{P}_1) \cap M(\hat{P}_2)) = \mu(M(\hat{P}_1))\mu(M(\hat{P}_2)) \quad \forall \hat{P}_1, \hat{P}_2 \in \check{\mathcal{L}}.$$

Proof: See (Lücke, 1996, Proof of Lemma 2.3). ■

Now we should no longer be surprised¹⁴ if, in orthodox quantum theory, we encounter quantum peculiarities such as¹⁵

$$\omega(\hat{P}) = 1 \not\implies (\omega(\hat{P} \wedge \hat{P}') = \omega(\hat{P}') \quad \forall \hat{P}' \in \mathcal{L}) \quad (1.7)$$

or¹⁶

$$\hat{P}_1 \wedge \hat{P}_2 = 0 \not\implies \hat{P}_1 \preceq \neg\hat{P}_2. \quad (1.8)$$

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¹²Recall Footnote 10.

¹³An example for the latter is given by the STERN-GERLACH experiment, where \hat{P} corresponds to ‘spin up’ and $\neg\hat{P}$ to ‘spin down’.

¹⁴According to Lemma 1.1.2, the (quantum logical) relations between (equivalence classes of) tests may be just a consequence of the (experimental) restrictions on the set of ‘states’.

¹⁵See also (Szabó, 1996, Sect. 3.1).

¹⁶By (1.5), $\hat{P}_1 \wedge \hat{P}_2 = 0$ means that there is no preparable property guaranteeing both $E_{\hat{P}_1}$ and $E_{\hat{P}_2}$.

Nevertheless, *simple quantum reasoning* according to the following rules is consistent:

- Choose a **classical** sublogic $(\mathcal{L}_c, \preceq, \neg)$ of $(\mathcal{L}, \preceq, \neg)$ and forget about all the other elements of \mathcal{L} .
- Then imagine that every **individual** – in whatever situation – has either property $E_{\hat{P}}$ or $E_{\neg\hat{P}}$ **if** $\hat{P} \in \mathcal{L}_c$.
- For $\omega \in \mathcal{S}$, imagine that $\omega(\hat{P})$ is the relative number of individuals having property $E_{\hat{P}}$ in an ensemble corresponding to ω **if** $\hat{P} \in \mathcal{L}_c$.
- Imagine that

- \preceq corresponds to common sense logical implication,
- \neg corresponds to common sense logical negation,
- \wedge corresponds to common sense logical ‘and’,
- \vee corresponds to common sense logical ‘or’.

This way all quantum peculiarities are avoided. For instance, in spite of (1.7), we may conclude

$$\left. \begin{array}{l} \omega(\hat{P}_1) = 1, \\ \hat{P}_1 \text{ compatible with } \hat{P}_2 \end{array} \right\} \implies \omega(\hat{P}_1 \wedge \hat{P}_2) = \omega(\hat{P}_2) \forall \omega$$

or even

$$\begin{aligned} & \hat{P}_1 \text{ compatible}^{17} \text{ with } \hat{P}_2 \\ \implies & \omega(\hat{P}_1 \vee \hat{P}_2) = \omega(\hat{P}_1 \wedge \neg\hat{P}_2) + \omega(\neg\hat{P}_1 \wedge \hat{P}_2) + \omega(\hat{P}_1 \wedge \hat{P}_2) \forall \omega \end{aligned}$$

— if $\hat{P}_1, \hat{P}_2 \in \mathcal{L}_c$.

Simple quantum reasoning naturally leads to the notion of *observable*.¹⁸

Definition 1.1.3 An *observable* A of a physical system modeled by the logic $(\mathcal{L}, \preceq, \neg)$ is a σ -morphism \hat{E}_A of the BOREL ring on the real line¹⁹ into $(\mathcal{L}, \preceq, \neg)$ which is unitary, i.e. $\hat{E}_A(\mathbb{R}) = \hat{1}$. It is called **bounded** iff $\hat{E}_A(\Delta) = 1$ for suitable compact $\Delta \in \mathbb{R}$.

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¹⁷For compatible \hat{P}_1, \hat{P}_2 :

$$\begin{aligned} \hat{P}_1 \vee \hat{P}_2 = (\hat{P}_1 \vee \hat{P}_2) \wedge (\neg\hat{P}_2 \vee \hat{P}_2) &= (\hat{P}_1 \wedge \neg\hat{P}_2) \vee \hat{P}_2 \\ &= (\hat{P}_1 \wedge \neg\hat{P}_2) \vee (\neg\hat{P}_1 \wedge \hat{P}_2) \vee (\hat{P}_1 \wedge \hat{P}_2) \end{aligned}$$

¹⁸The BOREL ring on \mathbb{R}^1 could be replaced by an arbitrary classical logic; possibly associated with some physical dimension.

¹⁹The BOREL *sets* on a locally compact space X form the smallest family of sets containing all compact subsets of X and being closed with respect to forming relative complements and countable unions.

The physical interpretation of \hat{E}_A in the sense of quantum reasoning is as follows:

Given $\omega \in \mathcal{S}$ and a BOREL subset Δ of \mathbb{R}^1 then $\omega(\hat{E}_A(\Delta))$ can be imagined as the relative number of individuals, in an ensemble corresponding to ω , for which $A \in \Delta$.

Consequently, the expectation value²⁰ for A in an ensemble corresponding to ω is given by

$$\bar{A}(\omega) = \int \lambda d\mu_\omega^A(\lambda). \quad (1.9)$$

with BOREL measure

$$\mu_\omega^A(B) \stackrel{\text{def}}{=} \omega(\hat{E}_A(B)) \quad \text{for BOREL subsets } B \subset \mathbb{R}^1. \quad (1.10)$$

Simple quantum reasoning can be applied to a whole family of observables A_1, A_2, \dots if and only if all the pairs

$$\left(\hat{E}_{A_j}(\Delta_j), \hat{E}_{A_k}(\Delta_k) \right), \quad \Delta_j, \Delta_k \in \mathbb{R}$$

are compatible.²¹

In order to make predictions for multiple tests one has to know how states change as a result of a simple test. Here we assume²²

LÜDERS *postulate*: For every $\hat{P} \in \mathcal{L}$ there is at least one corresponding ***measurement of first kind***, i.e. a simple test T with $[T] = \hat{P}$ causing a transition²³ $\omega \mapsto \omega_{,\hat{P}}$ whenever the result is ‘yes’. Here, if $\omega(\hat{P}) > 0$, $\omega_{,\hat{P}} \in \mathcal{S}$ is assumed to be uniquely characterized by the condition

$$\omega_{,\hat{P}}(\hat{P}') = \omega(\hat{P}')/\omega(\hat{P}) \quad \forall \hat{P}' \preceq \hat{P}.$$

Remark: Aktual measurements of first kind can be implemented using, e.g., CNOT-gates; see Footnote 23 of (Lücke, qip).

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²⁰Of course, the expectation value may be infinite!

²¹In a logic $(\mathcal{L}, \preceq, \neg)$ the sublogic generated by $\hat{P}_1, \hat{P}_2, \dots \in \mathcal{L}$ is classical if and only if all the pairs (\hat{P}_j, \hat{P}_k) are compatible (Piron, 1976, §2-2).

²²A first kind measurement corresponding to \hat{P} does not destroy any of the properties $E_{\hat{P}'}$ with \hat{P}', \hat{P} compatible (Lücke, 1996). Usually, a test causes a much more drastic change of the state or even ends by absorbing the corresponding individual. Intermediate measurements of first kind, typically, would be approximately realized by means of highly efficient filters.

²³Naively interpreted, $\omega_{,\hat{P}}(\hat{P}')$ describes the *conditional probability* in the state ω for $E_{\hat{P}'}$ – defined by (1.6) – being true provided $E_{\hat{P}}$ is true. In ordinary quantum theory the new statistical operator $\hat{T}_{\omega_{,\hat{P}}}$ is given by $\hat{P}\hat{T}_\omega\hat{P}/\omega(\hat{P})$.

By LÜDERS' postulate,²⁴ given the initial state $\omega \in \mathcal{S}$, the probability for the **homogeneous history** $(\hat{P}_1, \dots, \hat{P}_n)$ – i.e. for getting the answer ‘yes’ for all subsequent first kind measurements of a series corresponding to $\hat{P}_1, \dots, \hat{P}_n \in \mathcal{L}$ – should be²⁵

$$\omega(\hat{P}_1)\omega_{,\hat{P}_1}(\hat{P}_2) \cdots \omega_{,\hat{P}_1, \dots, \hat{P}_{n-1}}(\hat{P}_n).$$

Consistent quantum reasoning with respect to histories leads to the modern notion of **decoherent histories**.

Given a history $(\hat{P}_1, \dots, \hat{P}_n)$ not corresponding to a simple test, we can no longer be sure that there is an initial state for which $(\hat{P}_1, \dots, \hat{P}_n)$ is certain, i.e., for which $\omega(\hat{P}_1)\omega_{,\hat{P}_1}(\hat{P}_2) \cdots \omega_{,\hat{P}_1, \dots, \hat{P}_{n-1}}(\hat{P}_n) = 1$. Therefore the ‘logic’ of histories is weaker than that for simple tests and may provide a useful basis for generalizing quantum theory (Isham, 1995).

1.1.4 Symmetries and Dynamics

Just for simplicity we always use the following assumption, fulfilled in ordinary quantum theory:

$$\boxed{\mathcal{S} = \mathcal{S}_{(\mathcal{L}, \preceq, \neg)} \stackrel{\text{def}}{=} \text{set of all probability measures on } (\mathcal{L}, \preceq, \neg).} \quad (1.11)$$

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²⁴In the relativistic theory LÜDERS' postulate causes interesting problems (Schlieder, 1971) (see also (Mittelstaedt, 1983), (Mittelstaedt and Stachow, 1983)).

²⁵Naively interpreted, $\omega(\hat{P}_1)\omega_{,\hat{P}_1}(\hat{P}_2) \cdots \omega_{,\hat{P}_1, \dots, \hat{P}_{n-1}}(\hat{P}_n)$ is the probability for joint validity of the properties $E_{\hat{P}_1}, \dots, E_{\hat{P}_n}$ in the state ω . Usually (see, e.g., (Omnès, 1994), (Griffith, 1995)), unfortunately, this is formulated in the SCHRÖDINGER picture, thus imposing unnecessary restrictions.

Definition 1.1.4 A **symmetry** of a physical system modeled by the logic $(\mathcal{L}, \preceq, \neg)$ is an automorphism of $(\mathcal{L}, \preceq, \neg)$, i.e. a bijection of \mathcal{L} onto itself preserving the least upper bound and the orthocomplementation. A **dynamical semi-group** for such a system is a family $\{\alpha_t\}_{t \in \mathbb{R}_+}$ of symmetries α_t fulfilling the following three conditions:²⁶

- (i) $\alpha_0(\hat{P}) = \hat{P} \quad \forall \hat{P} \in \mathcal{L}$,
- (ii) $\alpha_{t_1} \circ \alpha_{t_2} = \alpha_{t_1+t_2} \quad \forall t_1, t_2 \in \mathbb{R}_+ \stackrel{\text{def}}{=} \{t \in \mathbb{R} : t \geq 0\}$,
- (iii) $t \mapsto \alpha_t$ is **weakly continuous**, i.e., for fixed $\hat{P} \in \mathcal{L}$ and $\omega \in \mathcal{S}$ the probability $\omega(\alpha_t(\hat{P}))$ is a **continuous** function of $t \in \mathbb{R}_+$.

The most important symmetries are the **time-translations** α_t , $t \in \mathbb{R}_+$, for systems which are homogeneous in time:

Let T be a macroscopic prescription for performing a simple test corresponding to $\hat{P} \in \mathcal{L}$. Then the prescription T_t to do everything prescribed by T just with time delay t characterizes a test corresponding to $\alpha_t(\hat{P})$.

Then the family $\{\alpha_t\}_{t \in \mathbb{R}_+}$ of time-translations, determining the **dynamics** of the system, is naturally assumed to be a dynamical semi-group.

1.2 Orthodox Quantum Mechanics and Algebraic Formulation

1.2.1 Logic and Observables

In ‘pioneer quantum mechanics’ (Primas, 1981) (without superselection rules²⁷) the logic $(\mathcal{L}, \preceq, \neg)$ described in Section 1.1.2 is realized as follows (**standard quantum logic**):

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²⁶If $\mathcal{S} \neq \mathcal{S}_{(\mathcal{L}, \preceq, \neg)}$ one should also require $\alpha^*(\mathcal{S}) = \mathcal{S}$ for the dual α^* of a symmetry α with respect to \mathcal{S} , defined by

$$(\alpha^*\omega)(\hat{P}) \stackrel{\text{def}}{=} \omega(\alpha(\hat{P})) \quad \forall \omega \in \mathcal{S}, \hat{P} \in \mathcal{L}.$$

Then the inverse of a symmetry need not be a symmetry.

²⁷A system modeled by $(\mathcal{L}, \preceq, \neg)$ is said to possess **superselection rules** if the **center**

$$\mathcal{C}_{(\mathcal{L}, \preceq, \neg)} \stackrel{\text{def}}{=} \left\{ \hat{P} \in \mathcal{L} : (\hat{P}, \hat{P}') \text{ compatible } \forall \hat{P}' \in \mathcal{L} \right\}$$

of $(\mathcal{L}, \preceq, \neg)$ is nontrivial ($\mathcal{L} \neq \mathcal{C} \neq \{\hat{0}, \hat{1}\}$). $(\mathcal{L}, \preceq, \neg)$ is called **irreducible** if $\mathcal{C} = \{\hat{0}, \hat{1}\}$.

- \mathcal{L} is given as the set of all projection operators²⁸ in some separable complex Hilbert space \mathcal{H} of dimension ≥ 2 .
- For arbitrary $\hat{P}_1, \hat{P}_2 \in \mathcal{L}$ we have

$$\begin{aligned} \hat{P}_1 \preceq \hat{P}_2 &\stackrel{\text{def}}{\iff} \hat{P}_1 \leq \hat{P}_2 \\ &\iff \left(\langle \Psi | \hat{P}_1 \Psi \rangle \leq \langle \Psi | \hat{P}_2 \Psi \rangle \quad \forall \Psi \in \mathcal{H} \right). \end{aligned}$$

- For every $\hat{P} \in \mathcal{L}$ we have

$$\neg \hat{P} \stackrel{\text{def}}{=} \hat{1} - \hat{P}.$$

Then, if $\dim(\mathcal{H}) \geq 3$, GLEASON's theorem (Gleason, 1957) tells us that for every $\omega \in \mathcal{S}(\mathcal{L}, \preceq, \neg)$ there is a unique positive trace class operators²⁹ $\hat{T}_\omega \in \mathcal{L}(\mathcal{H})$ fulfilling

$$\omega(\hat{P}) = \text{Tr}(\hat{T}_\omega \hat{P}) \quad \forall \hat{P} \in \mathcal{L}.$$

Every positive trace class operator \hat{T} of trace 1 can be written in the form

$$\hat{T} = \sum_{\nu=0}^{\infty} \underbrace{\lambda_\nu}_{\geq 0} \hat{P}_{\Psi_\nu}, \quad \sum_{\nu=0}^{\infty} \lambda_\nu = 1, \quad \mathcal{H} \ni \Psi_\nu \neq 0 \quad \forall \nu.$$

Here we use the standard notation

$$\hat{P}_{\Psi\Phi} \stackrel{\text{def}}{=} \left\langle \frac{\Psi}{\|\Psi\|} \middle| \Phi \right\rangle \frac{\Psi}{\|\Psi\|} \quad \forall \Phi \in \mathcal{H}, \Psi \in \mathcal{H} \setminus \{0\}.$$

Hence

$$\omega_\Psi(\hat{P}) \stackrel{\text{def}}{=} \text{Tr}(\hat{P}_{\Psi\hat{P}}) = \left\langle \frac{\Psi}{\|\Psi\|} \middle| \hat{P} \frac{\Psi}{\|\Psi\|} \right\rangle \quad \forall \Psi \in \mathcal{H} \setminus \{0\}, \hat{P} \in \mathcal{L}.$$

Now we have the following form of the LÜDERS postulate:

For every $\hat{P} \in \mathcal{L}$ there is at least one corresponding measurement of first kind, i.e. a simple test T with $[T] \hat{=} \hat{P}$ causing a transition $\omega \mapsto \omega_{,\hat{P}}$ whenever the result is 'yes', where

$$\hat{T}_{\omega_{,\hat{P}}} = \frac{\hat{P} \hat{T}_\omega \hat{P}}{\text{Tr}(\hat{P} \hat{T}_\omega \hat{P})}.$$

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²⁸Their specific physical identification depends on the *dynamics*, as discussed in (Mielnik, 1974) and (Lücke, 1995).

²⁹Conversely, every trace class operator of trace 1 induces a probability measure on standard quantum logic.

Especially, if $\omega = \omega_\Psi$ for some $\Psi \in \mathcal{H} \setminus \{0\}$, a first kind measurement corresponding to \hat{P}_Φ , $\Phi \in \mathcal{H} \setminus \{0\}$, causes the transition $\omega_\Psi \longrightarrow \omega_\Phi$ whenever the result is ‘yes’:

$$\omega_\Psi(\hat{P}_\Phi) > 0 \implies \frac{\hat{P}_\Phi \hat{P}_\Psi \hat{P}_\Phi}{\text{Tr}(\hat{P}_\Phi \hat{P}_\Psi \hat{P}_\Phi)} = \hat{P}_\Phi.$$

The corresponding *transition probability* is

$$\omega_\Psi(\hat{P}_\Phi) = \left| \left\langle \frac{\Phi}{\|\Phi\|} \middle| \frac{\Psi}{\|\Psi\|} \right\rangle \right|^2.$$

Exercise 1 Prove the following.³⁰

$$\hat{P}_1 \preceq \hat{P}_2 \iff \hat{P}_1 \mathcal{H} \subset \hat{P}_2 \mathcal{H}, \quad (1.12)$$

$$\hat{P}_1 \preceq \hat{P}_2 \iff \hat{P}_1 = \hat{P}_2 \hat{P}_1, \quad (1.13)$$

$$\neg \hat{P} \mathcal{H} = \mathcal{H} \ominus \hat{P} \mathcal{H} \stackrel{\text{def}}{=} \{\Psi \in \mathcal{H} : \langle \Phi | \Psi \rangle = 0 \text{ for all } \Phi \in \hat{P} \mathcal{H}\}, \quad (1.14)$$

$$\hat{P}_1 \wedge \hat{P}_2 \wedge \dots \stackrel{\text{def}}{=} \inf\{\hat{P}_1, \hat{P}_2, \dots\} = \text{orthogonal projection onto } \bigcap_{j=1}^{\infty} \hat{P}_j \mathcal{H}, \quad (1.15)$$

$$\sup\{\hat{P}_1, \hat{P}_2, \dots\} = \text{orthogonal projection onto } \overline{\text{span} \left(\bigcup_{j=1}^{\infty} \hat{P}_j \mathcal{H} \right)}, \quad (1.16)$$

$$\hat{P}_1 \wedge \hat{P}_2 = s\text{-}\lim_{n \rightarrow \infty} (\hat{P}_1 \hat{P}_2)^n \stackrel{\text{i.a.}}{\neq} \hat{P}_1 \hat{P}_2, \quad (1.17)$$

$$(\hat{P}_1, \hat{P}_2) \text{ compatible} \iff \hat{P}_1 \hat{P}_2 = \hat{P}_2 \hat{P}_1, \quad (1.18)$$

$$\begin{aligned} \hat{P}_1 \text{ Atom} &\stackrel{\text{def}}{\iff} (\hat{P}_1 \neq 0 \text{ and } \hat{P} \preceq \hat{P}_1 \implies \hat{P} \in \{0, \hat{P}_1\}) \\ &\iff \hat{P}_1 \mathcal{H} \text{ 1-dimensional,} \end{aligned} \quad (1.19)$$

$$\hat{P}_1 \wedge \neg \hat{P}_2 = 0 \text{ and } \hat{P}_1 \text{ Atom} \stackrel{\text{“covering law”}}{\implies} (\hat{P}_1 \vee \neg \hat{P}_2) \wedge \hat{P}_2 \text{ Atom}, \quad (1.20)$$

$$\omega_\Psi(\hat{P}) > 0 \implies \frac{\hat{P} \hat{P}_\Psi \hat{P}}{\text{Tr}(\hat{P} \hat{P}_\Psi \hat{P})} = \frac{\hat{P}_{\hat{P}_\Psi}}{\text{Tr}(\hat{P}_{\hat{P}_\Psi})}. \quad (1.21)$$

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³⁰Assuming irreducibility, atomicity and (1.20) one may prove that – apart from some exceptional cases – $(\mathcal{L}, \preceq, \neg)$ is isomorphic to the logic of all projection operators on some *generalized HILBERT space* (Piron, 1976, Section 3–1). According to (1.21) the probability for the homogeneous history (\hat{P}_1, \hat{P}_2) is $\omega_\Psi(\hat{P}_2 \hat{P}_1 \hat{P}_2)$, where $\hat{0} \leq \hat{P}_2 \hat{P}_1 \hat{P}_2 \leq \hat{1}$ but $\hat{P}_2 \hat{P}_1 \hat{P}_2 \stackrel{\text{i.g.}}{\notin} \mathcal{L}$. Note (Davies, 1976, Lemma 2.4), however.

Here, according to Definition 1.1.3, an observable A corresponds to a **projection valued measure**, i.e. a mapping \hat{E}_A from the ring of BOREL sets (over \mathbb{R}^1) into \mathcal{L} such that:

$$\begin{aligned} (\text{PVM}_1) : \hat{E}_A(\mathbb{R}^1) &= \hat{1}, & (\text{Normalization}) \\ (\text{PVM}_1) : \hat{E}_A\left(\bigcup_{j=1}^{\infty} B_j\right) &= s\text{-}\lim_{n \rightarrow \infty} \sum_{j=1}^n \hat{E}_A(B_j), \\ &\text{whenever the } B_j \text{ are } \mathbf{mutually disjoint} \text{ BOREL sets.} & (\sigma\text{-Additivity}) \end{aligned}$$

\hat{E}_A gives rise to a self-adjoint operator \hat{A} , uniquely characterized by³¹

$$\bar{A}(\omega_\Psi) = \left\langle \frac{\Psi}{\|\Psi\|} \left| \hat{A} \frac{\Psi}{\|\Psi\|} \right. \right\rangle \quad \forall \Psi \in D_{\hat{A}} \setminus \{0\} \quad (1.22)$$

(remember (1.9)/(1.10)), where

$$D_{\hat{A}} = \left\{ \Psi : \int_{\lambda \in \mathbb{R}^1} \lambda^2 \omega_\Psi(\hat{E}_A(d\lambda)) < \infty \right\} \quad (1.23)$$

is the domain of \hat{A} (see e.g. (Achieser and Glasmann, 1965)). For this operator one usually writes

$$\hat{A} = \int_{\lambda \in \mathbb{R}^1} \lambda \hat{E}_A(d\lambda). \quad (1.24)$$

With this operator we also have

$$\begin{aligned} \bar{A}(\omega) &= \text{Tr} \left(\hat{T}_\omega \hat{A} \right) \quad \forall \omega \in \mathcal{S}_{(\mathcal{L}, \preceq, \cdot)} \\ &\text{if } \hat{A} \text{ is bounded} \end{aligned} \quad (1.25)$$

resp.

$$\begin{aligned} \bar{A}(\omega) &= \lim_{\Lambda_+ \rightarrow +\infty} \text{Tr} \left(\hat{T}_\omega \hat{A} \hat{E}_{\hat{A}}((0, \Lambda_+]) \right) \\ &+ \lim_{\Lambda_- \rightarrow -\infty} \text{Tr} \left(\hat{T}_\omega \hat{A} \hat{E}_{\hat{A}}((\Lambda_-, 0]) \right) \quad \forall \omega \in \mathcal{S}_{(\mathcal{L}, \preceq, \cdot)} \end{aligned}$$

if \hat{A} is unbounded, where the l.h.s. is defined iff the r.h.s is.

According to the well known **spectral!theorem** (see e.g. (Reed und Simon, 1972)), for every self-adjoint operator \hat{A} there is a unique regular³² projector-valued measure \hat{E}_A fulfilling (1.22)/(1.23), called the **spectral!measure** of \hat{A} . In this sense, according to Definition 1.1.3, the observables of orthodox quantum mechanics may be identified with the self-adjoint operators.³³

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³¹This is very useful for heuristic physical identification of observables.

³²A projector-valued measure \hat{E}_A is called **regular** if

$$\begin{aligned} \hat{E}_A(B) &= \sup \left\{ \hat{E}_A(C) : B \supset C \text{ compact} \right\} \\ &= \inf \left\{ \hat{E}_A(\mathcal{O}) : B \subset \mathcal{O} \text{ open} \right\} \end{aligned}$$

holds for **all** BOREL subsets B of \mathbb{R} (not only for $B = (-\infty, \lambda]$, $\lambda \in \mathbb{R}^1$).

³³We skip physical dimensions to allow for addition of these operators.

Exercise 2 Determine the spectral measures for the following operators of elementary $L^2(\mathbb{R}^1, dx)$ -quantum mechanics:

- (i) position operator,
- (ii) linear momentum operator,
- (iii) energy operator of the harmonic oscillator,
- (iv) zero operator,
- (v) identity operator.

Let \hat{A} be a self-adjoint operator on the complex HILBERT space \mathcal{H} with spectral measure \hat{E}_A and let f be some complex-valued BOREL function³⁴ on \mathbb{R}^1 . Then there is a unique operator $f(\hat{A})$, written

$$f(\hat{A}) = \int f(\lambda) \hat{E}_A(d\lambda), \quad (1.26)$$

with domain

$$D_{f(\hat{A})} \stackrel{\text{def}}{=} \left\{ \Psi \in \mathcal{H} : \int |f(\lambda)|^2 \omega_\Psi(\hat{E}_A(d\lambda)) < \infty \right\} \quad (1.27)$$

fulfilling

$$\left\langle \frac{\Psi}{\|\Psi\|} \middle| f(\hat{A}) \frac{\Psi}{\|\Psi\|} \right\rangle = \int f(\lambda) \omega_\Psi(\hat{E}_A(d\lambda)) \quad \forall \Psi \in D_{f(\hat{A})} \setminus \{0\}.$$

If f is real-valued then $f(\hat{A})$ is self-adjoint on this domain and its spectral measure is characterized by

$$\hat{E}_{f(\hat{A})}(J) = \hat{E}_A(f^{-1}(J)) \quad \text{for all intervals } J \subset \mathbb{R}. \quad (1.28)$$

If f is bounded then the operator $f(\hat{A})$ is bounded and

$$\|f(\hat{A})\| \stackrel{\text{def}}{=} \sup_{\Psi \in \mathcal{H} \setminus \{0\}} \left\| \hat{A} \frac{\Psi}{\|\Psi\|} \right\| \leq \sup_{\lambda \in \mathbb{R}^1} |f(\lambda)|. \quad (1.29)$$

Exercise 3 Prove the following:

$$f(\lambda) = f_0 \text{ for all } \lambda \in \mathbb{R}^1 \implies f(\hat{A}) = f_0 \hat{1}, \quad (1.30)$$

$$f(\lambda) = g(\lambda) + h(\lambda) \text{ for all } \lambda \in \mathbb{R}^1 \implies f(\hat{A}) = \overline{g(\hat{A}) + h(\hat{A})}, \quad (1.31)$$

$$f(\lambda) = g(\lambda)h(\lambda) \text{ for all } \lambda \in \mathbb{R}^1 \implies f(\hat{A}) = \overline{g(\hat{A})h(\hat{A})}, \quad (1.32)$$

$$g(f(\hat{A})) = h(\hat{A}) \text{ if } f \text{ is real-valued and } g(f(\hat{\lambda})) = h(\hat{\lambda}) \text{ for all } \lambda \in \mathbb{R}^1, \quad (1.33)$$

$$|f(\lambda)| = 1 \text{ for all } \lambda \in \mathbb{R}^1 \implies f(\hat{A}) \text{ unitary}, \quad (1.34)$$

$$\hat{E}_A(B) = \chi_B(\hat{A}) \text{ for all BOREL sets } B. \quad (1.35)$$

³⁴If B is a BOREL subset of \mathbb{C} then $f^{-1}(B)$ has to be a BOREL subset of \mathbb{R}^1 .

1.2.2 Symmetries and Dynamics

If \mathcal{H} is a complex HILBERT space, let us denote by $(\mathcal{L}_{\mathcal{H}}, \preceq, \neg)$ the standard quantum logic described in 1.2.1.

Theorem 1.2.1 (Wigner) *Let \mathcal{H} be a complex HILBERT space of dimension ≥ 3 . Then a map $\alpha : \mathcal{L}_{\mathcal{H}} \rightarrow \mathcal{L}_{\mathcal{H}}$ is a symmetry of $(\mathcal{L}_{\mathcal{H}}, \preceq, \neg)$ iff there is either a unitary or an anti-unitary operator \hat{U} with³⁵*

$$\alpha(\hat{P}) = \hat{U}\hat{P}\hat{U}^* \quad \forall \hat{P} \in \mathcal{L}_{\mathcal{H}}.$$

Proof: See (Piron, 1976, §3-2). ■

For the elements of a dynamical semi-group of $(\mathcal{L}_{\mathcal{H}}, \preceq, \neg)$ the choice of anti-unitary \hat{U} is excluded:

Theorem 1.2.2 *Let \mathcal{H} be a complex HILBERT space of dimension ≥ 3 and let $\{\alpha_t\}_{t \in \mathbb{R}_+}$ be a dynamical semi-group of $(\mathcal{L}_{\mathcal{H}}, \preceq, \neg)$. Then there is a unique self-adjoint operator \hat{H} fulfilling*

$$\alpha_t(\hat{P}) = e^{\frac{i}{\hbar}\hat{H}t}\hat{P}e^{-\frac{i}{\hbar}\hat{H}t} \quad \forall \hat{P} \in \mathcal{L}_{\mathcal{H}}, t \in \mathbb{R}_+.$$

Proof: See (Lücke, 1996, Sect. 3.3). ■

1.2.3 Algebras of Bounded Observables

Unbounded observables \hat{A} have the unpleasant feature that their domain is always smaller than \mathcal{H} , by the HELLINGER-TOEPLITZ theorem (see (Reed und Simon, 1972, corollary to the Closed Graph Theorem II.12).) This causes lots of technical complications. Fortunately, from the principal point of view, it is sufficient to know the spectral operators $\hat{E}_A(J)$, which are always bounded, for all intervals $J \subset \mathbb{R}$. This allows for taking advantage of the powerful mathematical theory of algebras of bounded operators.

The set $\mathcal{L}(\mathcal{H})$ of **all** bounded operators \hat{A} in \mathcal{H} (with $D_{\hat{A}} = \mathcal{H}$) is a complex BANACH **algebra**, i.e. a complex BANACH space³⁶ with associative³⁷ and distributive multiplication³⁸ fulfilling the so-called **product inequality**

$$\|\hat{A}\hat{B}\| \leq \|\hat{A}\| \|\hat{B}\|.$$

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³⁵Especially, we have $\alpha(\hat{P}_{\Phi}) = \hat{P}_{\hat{U}\Phi}$.

³⁶In this case the set of all bounded operators with ordinary addition, multiplication by complex numbers, and operator norm.

³⁷This means: $\hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}$ for all elements $\hat{A}, \hat{B}, \hat{C}$ of the Algebra, and $(\alpha\beta)(\hat{A}\hat{B}) = (\alpha\hat{A})(\beta\hat{B})$ for all complex Numbers α, β and all elements $\hat{A}, \hat{B}, \hat{C}$ of the Algebra.

³⁸In this case ordinary multiplication with operators and/or numbers.

Transition to the adjoint operator is a *involution*, i.e. a mapping $\hat{A} \rightarrow \hat{A}^*$ fulfilling the following three conditions:

$$\begin{aligned} (I_1): & \quad (\hat{A}^*)^* = \hat{A} \\ (I_2): & \quad (\hat{A}\hat{B})^* = \hat{B}^*\hat{A}^* \\ (I_3): & \quad (\alpha\hat{A} + \beta\hat{B})^* = \bar{\alpha}\hat{A}^* + \bar{\beta}\hat{B}^* \end{aligned}$$

$\mathcal{L}(\mathcal{H})$ is also a *C*-algebra*³⁹, i.e. a complex BANACH algebra with involution $*$, obeying the condition

$$\|\hat{A}^*\hat{A}\| = \|\hat{A}\|^2.$$

Exercise 4 Proof that $\|\hat{A}^*\| = \|\hat{A}\|$, hence also $\|\hat{A}\hat{A}^*\| = \|\hat{A}\|^2$, holds for every C*-algebra.

$\mathcal{L}(\mathcal{H})$ is even a VON NEUMANN *algebra*, i.e. a subalgebra \mathcal{M} of $\mathcal{L}(\mathcal{H})$, \mathcal{H} some complex HILBERT space, that is given by the *commutant*

$$\mathcal{N}' \stackrel{\text{def}}{=} \left\{ \hat{A} \in \mathcal{L}(\mathcal{H}) : [\hat{A}, \hat{B}]_- = 0 \forall \hat{B} \in \mathcal{N} \right\}$$

of some $*$ -invariant subset $\mathcal{N} \subset \mathcal{L}(\mathcal{H})$:

$$\mathcal{M} = (\mathcal{N} \cup \mathcal{N}^*)' \quad (\text{and hence } \mathcal{M} = \mathcal{M}'').$$

Exercise 5 Let \mathcal{B} be a set and r a binary relation on \mathcal{B} . Show that

$$\mathcal{A}_1 \subset \mathcal{A}_2 \implies \mathcal{A}_1^r \supset \mathcal{A}_2^r$$

holds for all subsets $\mathcal{A}_1, \mathcal{A}_2$ of \mathcal{B} , where

$$\mathcal{A}^r \stackrel{\text{def}}{=} \left\{ \hat{B} \in \mathcal{B} : r(\hat{B}, \hat{A}) \forall \hat{A} \in \mathcal{A} \right\} \quad \forall \mathcal{A} \subset \mathcal{B}.$$

Moreover, show for **symmetric** r that $\mathcal{A} \subset \mathcal{A}^{rr}$ and therefore also

$$\mathcal{A}^{rrr} = \mathcal{A}^r$$

holds for all $\mathcal{A} \subset \mathcal{B}$.

Another important BANACH algebra with involution is the set $\mathcal{T}_1 \subset \mathcal{L}(\mathcal{H})$ of trace class operators with the norm

$$\|\hat{A}\|_{\text{Tr}} \stackrel{\text{def}}{=} \text{Tr} \sqrt{\hat{A}^*\hat{A}} \geq \|\hat{A}\|.$$

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³⁹Called a *completely regular algebra* by NEUMARK (Neumark, 1959).

Identifying $\hat{A} \in \mathcal{L}(\mathcal{H})$ with the mapping

$$\hat{T} \rightarrow \text{Tr}(\hat{T}\hat{A})$$

we get

$$\mathcal{L}(\mathcal{H}) = \mathcal{T}_1(\mathcal{H})^* \stackrel{\text{def}}{=} \{ \text{linear continuous mappings } \mathcal{T}_1(\mathcal{H}) \rightarrow \mathbf{C} \} \quad (1.36)$$

(see (Bratteli and Robinson, 1979, Proposition 2.4.3)). Similarly,⁴⁰ we have

$$\mathcal{T}_1(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})^* \quad (1.37)$$

((Gaal, 1973, pp 100/101)) by this identification, where $\mathcal{C}_1(\mathcal{H})$ denotes the C^* -subalgebra⁴¹ of $\mathcal{L}(\mathcal{H})$ consisting of all compact (=completely continuous) operators⁴² in \mathcal{H} .

1.2.4 State Functionals

According to GLEASON'S theorem, if \mathcal{H} is separable and of dimension > 2 , every probability measure on the standard quantum logic is the restriction (to the projection operators) of a (unique) mapping $\omega : \mathcal{L}(\mathcal{H}) \rightarrow \mathbf{C}$ of the form

$$\omega(\hat{A}) = \text{Tr}(\hat{T}_\omega \hat{A}), \quad \hat{T}_\omega \in \mathcal{T}_1(\mathcal{H}), \quad \text{Tr}(\hat{T}_\omega) = 1, \quad \hat{T}_\omega = \hat{T}_\omega^* \geq 0. \quad (1.38)$$

Exercise 6

- (i) Given $\hat{P}_1, \dots, \hat{P}_n \in \mathcal{L}_{\mathcal{H}}$ and $\Psi \in \mathcal{H} \setminus \{0\}$, show that the probability for the homogeneous history $(\hat{P}_1, \dots, \hat{P}_n)$ in a state prepared according to ω_Ψ is

$$\frac{\|\hat{P}_n \cdots \hat{P}_1 \Psi\|^2}{\|\Psi\|^2} = \omega_\Psi(\hat{P}_1 \cdots \hat{P}_n \hat{P}_{n-1} \cdots \hat{P}_1)$$

and that

$$\hat{0} \leq \hat{P}_n \cdots \hat{P}_1 \hat{P}_{n-1} \cdots \hat{P}_1 \leq \hat{1}$$

even though $\hat{P}_n \cdots \hat{P}_1 \hat{P}_{n-1} \cdots \hat{P}_1 \stackrel{\text{i.g.}}{\notin} \mathcal{L}_{\mathcal{H}}$.

⁴⁰Note that (1.36) resp. (1.37) is analogous to $l^\infty = (l^1)^*$, resp. $l^1 = (c_0)^*$, where: $l^\infty \stackrel{\text{def}}{=} \{ \text{bounded sequences} \}$, $c_0 \stackrel{\text{def}}{=} \{ \text{null sequences} \}$, $l^1 \stackrel{\text{def}}{=} \{ \text{absolutely convergent sequences} \}$

⁴¹See (Gaal, 1973, Theorem 3 and Proposition 4 of Section II.2)

⁴²Compact operators are uniform limits of increasing countable sequences of finite rank operators (Gaal, 1973, Theorem 7 of Section II.2).

- (ii) Show that, contrary to $\mathcal{L}_{\mathcal{H}}$, the set $\{\hat{F} \in \mathcal{L}(\mathcal{H}) : \hat{0} \leq \hat{F} \leq \hat{1}\}$ of all **effects** with its natural semi-ordering and ‘orthocomplementation’ is **not** a logic.⁴³

The modern notion of state is as follows:⁴⁴ A **state** on a C^* -algebra \mathcal{A} with unit is a mapping $\hat{A} \rightarrow \omega(\hat{A})$ of \mathcal{A} into the **complex** numbers fulfilling the following three conditions for all $\hat{A}, \hat{B} \in \mathcal{A}$ and $\alpha \in \mathbb{C}$:

$$\begin{aligned} (S_1) : \quad & \omega(\hat{A} + \alpha\hat{B}) = \omega(\hat{A}) + \alpha\omega(\hat{B}) && \text{(linearity)} \\ (S_2) : \quad & \omega(\hat{1}) = 1 && \text{(normalization)} \\ (S_3) : \quad & \omega(\hat{A}^*\hat{A}) \geq 0 && \text{(positivity)}^{45} \end{aligned}$$

Exercise 7 Show that the following three conditions are fulfilled for every state ω on a C^* -algebra with unit element:

$$\begin{aligned} (i) \quad & |\omega(\hat{A}^*\hat{B})|^2 \leq \omega(\hat{A}^*\hat{A})\omega(\hat{B}^*\hat{B}) && \text{(CAUCHY SCHWARZ inequality)} \\ (ii) \quad & \omega(\hat{A}^*) = \overline{\omega(\hat{A})} && \text{(hermiticity)} \\ (iii) \quad & \omega(\hat{A}_n) \rightarrow \omega(\hat{A}) \quad \text{if } \|\hat{A} - \hat{A}_n\| \rightarrow 0 && \text{(continuity)} \end{aligned}$$

However, due to $\mathcal{T}_1(\mathcal{H}) = \mathcal{C}(\mathcal{H})^*$, there are also so-called *singular* states,⁴⁶ for which $\omega(\hat{P}) = 0$ whenever \hat{P} has finite rank. Of course, the restrictions of such functionals do not define probability measures on the standard quantum logic since they cannot be σ -additive. Therefore the state functionals need an additional characterization which relies on the following.

Lemma 1.2.3 *Let \mathcal{M} be a VON NEUMANN subalgebra of $\mathcal{L}(\mathcal{H})$, let I be some ordered index set, and let $\{\hat{A}_i\}_{i \in I} \subset \mathcal{M}$ be an increasing net of positive operators with $\sup_{\mathbb{R}} \{\|\hat{A}_i\| : i \in I\} < \infty$. Then $\sup_{\mathcal{L}(\mathcal{H})} \{\hat{A}_i : i \in I\}$ exists and is an element of the algebra \mathcal{M} .*

Proof: See (Bratteli and Robinson, 1979, Lemma 2.4.19). ■

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⁴³**Hint:** Show that an effect \hat{F} is a projection if and only if $\hat{0} = \inf_{\{\text{effects}\}} \{\hat{F}, (\hat{1} - \hat{F})\}$.

⁴⁴This way explicit use of the HILBERT space becomes unnecessary, in principle (see Sect. 1.3.3 for details).

⁴⁵It would be quite tedious to show in general for C^* -algebras that $\hat{A}^*\hat{A} = -\hat{B}^*\hat{B} \implies \hat{A}^*\hat{A} = 0$.

⁴⁶In general, a state ω on VON NEUMANN algebra is called **singular**, iff for every nonzero projection operator \hat{P} there is another nonzero projection operator \hat{P}' for which $\omega(\hat{P}') = 0$ and $\hat{P}' \preceq \hat{P}$.

Definition 1.2.4 A state ω on a VON NEUMANN algebra \mathcal{M} is said to be **normal**, iff

$$\omega \left(\sup_{\mathcal{L}(\mathcal{H})} \left\{ \hat{A}_i : i \in I \right\} \right) = \sup_{\mathbb{R}} \left\{ \omega(\hat{A}_i) : i \in I \right\}$$

holds for every net fulfilling the requirements of Lemma 1.2.3.

Theorem 1.2.5 Let \mathcal{M} be a VON NEUMANN subalgebra of $\mathcal{L}(\mathcal{H})$. Then the normal states ω on \mathcal{M} are exactly those of the form (1.38).

Proof: See (Bratteli and Robinson, 1979, Theorem 2.4.21). ■

Exercise 8 Prove Theorem 1.2.5 for $\mathcal{M} = \mathcal{L}(\mathcal{H})$, where \mathcal{H} is separable.⁴⁷

According to Theorem 1.2.5 (and GLEASON's theorem) the probability measures on the standard quantum logic correspond to restrictions of **normal** states on $\mathcal{L}(\mathcal{H})$ to the projection operators, if $\dim(\mathcal{H}) > 2$.

Definition 1.2.6 A state ω on a C^* -algebra \mathcal{A} is said to be **mixed**, iff there are states $\omega_1 \neq \omega_2$ on \mathcal{A} and a real number $\lambda \in (0, 1)$ fulfilling

$$\omega(\hat{A}) = \lambda\omega_1(\hat{A}) + (1 - \lambda)\omega_2(\hat{A}) \quad \text{for every } \hat{A} \in \mathcal{A}.$$

Otherwise ω is said to be **pure**.

Exercise 9

- (i) Show that the states ω_1, ω_2 of Definition 1.2.6 must both be **normal**, if ω is normal.
- (ii) Prove that a normal state ω on $\mathcal{L}(\mathcal{H})$ is pure iff there is a normed vector $\Omega \in \mathcal{H}$ fulfilling

$$\omega(\hat{A}) = \left\langle \Omega \left| \hat{A} \Omega \right. \right\rangle = \text{Tr}(\hat{P}_\Omega \hat{A}) \quad \text{for all } \hat{A} \in \mathcal{L}(\mathcal{H}).$$

⁴⁷Hint: If $\left\{ \hat{A}_i \right\}_{i \in I}$ is an increasing bounded net then so is $\left\{ \sqrt{\hat{T}} \hat{A}_i \sqrt{\hat{T}} \right\}_{i \in I}$.

1.3 Algebraic Formulation of General Quantum Theory⁴⁸

1.3.1 Partial States

If one is interested only in a certain subset⁴⁹ \mathcal{A} of physical entities A , it is sufficient to know the *partial state* $\omega = \omega_{\text{total}}/\mathcal{M}$ of the proper normal state ω_{total} on $\mathcal{L}(\mathcal{H}_{\text{total}})$ with respect to the smallest VON NEUMANN subalgebra \mathcal{M} of $\mathcal{L}(\mathcal{H}_{\text{total}})$ that contains all the projection operators $\hat{E}_A((-\infty, \lambda])$ with $A \in \mathcal{A}$, $\lambda \in \mathbb{R}^1$.

An immediate consequence of Theorem 1.2.5 is the following.

Corollary 1.3.1 *Let \mathcal{M}_2 be a VON NEUMANN subalgebra of the VON NEUMANN algebra \mathcal{M}_1 . Then the set of all normal states on \mathcal{M}_2 coincides with the set of all partial states of normal states on \mathcal{M}_1 .*

We conclude that in orthodox quantum theory the relevant states are the **normal** states on the VON NEUMANN subalgebra $\mathcal{M} \subset \mathcal{L}(\mathcal{H}_{\text{total}})$ of interest.

Exercise 10 Show that even if ω_{total} is pure the partial state $\omega = \omega_{\text{total}}/\mathcal{M}$ may be mixed.

From now on we consider only quantum logics $(\mathcal{L}, \preceq, \neg)$ of the following type.⁵⁰

There is a separable complex HILBERT space \mathcal{H} and a VON NEUMANN subalgebra \mathcal{M} of $\mathcal{L}(\mathcal{H})$ by which $(\mathcal{L}, \preceq, \neg)$ is realized in the following way:

- $\mathcal{L} = \mathcal{L}_{\mathcal{M}} \stackrel{\text{def}}{=} \{ \hat{P} \in \mathcal{M} : \hat{P}^* = \hat{P} = \hat{P}^2 \}$,
- $\hat{P}_1 \preceq \hat{P}_2 \stackrel{\text{def}}{\iff} \hat{P}_1 \leq \hat{P}_2 \quad \forall \hat{P}_1, \hat{P}_2 \in \mathcal{L}$,
- $\neg \hat{P} \stackrel{\text{def}}{=} \hat{1} - \hat{P} \quad \forall \hat{P} \in \mathcal{L}$.

Exercise 11 For bounded self-adjoint operators it is known that they commute (in the naive sense) if and only if all their spectral projections commute⁵¹ (Neumark, 1959,

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⁴⁸See (Roberts and Roepstorff, 1969).

⁴⁹A typical case would be the description of extended structures via their centers of mass.

⁵⁰That the projection operators of a VON NEUMANN algebra always form a sublogic $(\mathcal{L}_{\mathcal{M}}, \preceq, \neg)$ of the corresponding standard quantum logic follows from (1.17) (note $\hat{P}_1 \vee \hat{P}_2 = \neg(\neg \hat{P}_1 \wedge \neg \hat{P}_2)$) and Lemma 1.2.3. Necessary and sufficient conditions for a given quantum logic to be isomorphic to a sublogic of standard quantum logic are given in (Gudder, 1979).

⁵¹For unbounded self-adjoint operators the latter is the appropriate definition for commutativity (Reed und Simon, 1972, Sect. VIII.5).

Theorem VII of §17.4) (or (Riesz and Sz.-Nagy, 1982, Theorem on Page 335)). Use this to show that

$$\hat{A} = \hat{A}^* \in \mathcal{M} \implies \hat{E}_A(J) \in \mathcal{M}$$

holds for all $\hat{A} \in \mathcal{L}(\mathcal{H})$ and all intervals $J \subset \mathbb{R}$.

Theorem 1.3.2 (Generalized Gleason Theorem) *Let \mathcal{M} be a VON NEUMANN algebra with no type I_2 summand.⁵² Every finitely additive probability measure ω on $\mathcal{L}_{\mathcal{M}}$ can be extended to a state on \mathcal{M} . This state is normal if and only if the corresponding probability measure is completely additive.*

Proof: See (Maeda, 1989). ■

We conclude⁵³ that the physically relevant states will **always** be the **normal** states on the corresponding VON NEUMANN algebra \mathcal{M} .

Definition 1.3.3 *A ***-morphism** of a C^* -algebra \mathcal{A}_1 into a C^* -algebra \mathcal{A}_2 is a mapping γ of \mathcal{A}_1 into \mathcal{A}_2 respecting linearity, multiplication, and involution:*

$$\begin{aligned} (M_1) : \quad & \gamma(\hat{A} + \beta \hat{B}) = \gamma(\hat{A}) + \beta \gamma(\hat{B}) \\ (M_2) : \quad & \gamma(\hat{A} \hat{B}) = \gamma(\hat{A}) \gamma(\hat{B}) \\ (M_3) : \quad & \gamma(\hat{A}^*) = \gamma(\hat{A})^* \end{aligned}$$

*A ***-morphism** of a C^* -algebra \mathcal{A}_1 into a C^* -algebra \mathcal{A}_2 is called a ***-isomorphism** if it is a bijection (one-one and onto). A ***-automorphism** of a C^* -algebra \mathcal{A} is a ***-isomorphism** of \mathcal{A} onto itself.*

Now Theorem 1.3.2 has the following consequence (remember Definition 1.1.4):

Corollary 1.3.4 *Let \mathcal{M} be a VON NEUMANN algebra with no type I_2 summand and let $\{\alpha_t\}_{t \in \mathbb{R}}$ be a dynamical semi-group for some system modeled by $(\mathcal{L}_{\mathcal{M}}, \preceq, \neg)$. Then $\{\alpha_t\}_{t \geq 0}$ is the restriction to $\mathcal{L}_{\mathcal{M}}$ of a weakly* continuous⁵⁴ 1-parameter semi-group of *-automorphisms of \mathcal{M} .*

Proof: See (Lücke, 1996, Appendix A). ■

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⁵²The latter means, if \mathcal{M} is a VON NEUMANN subalgebra of $\mathcal{L}(\mathcal{H})$, that there is no projection $\hat{P} \in \mathcal{L}_{\mathcal{H}}$ onto a 2-dimensional subspace of \mathcal{H} for which $\mathcal{M} = \hat{P} \mathcal{L}(\mathcal{H}) \hat{P} + (\hat{1} - \hat{P}) \mathcal{M} (\hat{1} - \hat{P})$ (see (Neumark, 1959, §38 Nr. 3 Theorem 2)).

⁵³Remember our assumption (1.11).

⁵⁴ $\{\alpha_t\}_{t \geq 0}$ is **weakly* continuous** iff $\omega(\alpha_t(\hat{A}))$ is continuous in t for all normal states ω and all $\hat{A} \in \mathcal{M}$ (Bratteli and Robinson, 1979, Proposition 2.4.3).

In the HAAG-DOPLICHER-ROBERTS theory (see (Haag, 1992)) the relevant VON NEUMANN algebra \mathcal{M} is to be constructed by weak closure of a suitable representation of some C^* -algebra. Therefore we have to discuss the latter concept.

Definition 1.3.5 *A representation of a C^* -algebra \mathcal{A} in the complex HILBERT space \mathcal{H} is a $*$ -morphism π of \mathcal{A} into $\mathcal{L}(\mathcal{H})$. The representation is said to be **faithful** iff π is an injection.*

Theorem 1.3.6 *Let γ be a $*$ -morphism of the C^* -algebra \mathcal{A}_1 into the C^* -algebra \mathcal{A}_2 . Then $\gamma(\mathcal{A}_1)$ is a C^* -subalgebra of \mathcal{A}_2 and.⁵⁵*

$$\left\| \gamma(\hat{A}) \right\|_{\mathcal{A}_2} \leq \left\| \hat{A} \right\|_{\mathcal{A}_1} \quad \forall \hat{A} \in \mathcal{A}_1.$$

Proof: See (Bratteli and Robinson, 1979, Lemma 2.3.1) ((Dixmier, 1969, sections 1.3.7 and 1.8.3)). ■

Exercise 12 Prove Theorem 1.3.6 for the case that $\pi(\mathcal{A}_1)$ is known to be a C^* -subalgebra of \mathcal{A}_2 .

An immediate consequence of Theorem 1.3.6 is the following

Corollary 1.3.7 *Let π be a representation of the C^* -algebra \mathcal{A} in \mathcal{H} . Then $\pi(\mathcal{A})$ is a C^* -subalgebra of $\mathcal{L}(\mathcal{H})$. If the representation is faithful, then:*

$$\left\| \pi(\hat{A}) \right\|_{\mathcal{L}(\mathcal{H})} = \left\| \hat{A} \right\|_{\mathcal{A}} \quad \forall \hat{A} \in \mathcal{A}.$$

Warning: Even if \mathcal{A} is a VON NEUMANN algebra and π is **faithful** it may happen⁵⁶ that $\sup_{\mathcal{L}(\mathcal{H})} \left\{ \pi(\hat{A}_i) : i \in I \right\} \notin \pi(\mathcal{A})$ for some net $\{\hat{A}_i\}_{i \in I} \subset \mathcal{A}$ of the type considered in Lemma 1.2.3. Then $\pi(\mathcal{A})$ is not a VON NEUMANN subalgebra of $\mathcal{L}(\mathcal{H})$ and π will not map $(\mathcal{L}_{\mathcal{A}}, \preceq, \neg)$ onto a sublogic of $(\mathcal{L}, \preceq, \neg)$.

⁵⁵Thus application to the special case $\pi = \text{identity}$ shows that the norm of a C^* -algebra is uniquely fixed by the algebraic structure.

⁵⁶Just consider the GNS representation π_{ω} (Theorem 1.3.8) for the mixture ω of a separating state ω on $\mathcal{L}(\mathcal{H})$ (see (Bratteli and Robinson, 1979, Prop. 2.5.6)) with a non-normal state on $\mathcal{L}(\mathcal{H})$ (recall Statement (i) of Exercise 9).

1.3.2 GNS-Representation

Theorem 1.3.8 *Let \mathcal{A} be a C^* -algebra with $\hat{1}$ and let ω be a state on \mathcal{A} . Then the set of equivalence classes*

$$[\hat{A}_1] \stackrel{\text{def}}{=} \{ \hat{A}_2 \in \mathcal{A}, \omega((\hat{A}_1 - \hat{A}_2)^*(\hat{A}_1 - \hat{A}_2)) = 0 \}, \hat{A}_1 \in \mathcal{A}$$

with linear structure

$$\alpha [\hat{A}] + \beta [\hat{B}] \stackrel{\text{def}}{=} [\alpha \hat{A} + \beta \hat{B}]$$

and inner product

$$\langle [\hat{A}] | [\hat{B}] \rangle \stackrel{\text{def}}{=} \omega(\hat{A}^* \hat{B})$$

is a complex pre-HILBERT space. Moreover, continuous extension of the operators

$$\pi_\omega(\hat{A})[\hat{B}] \stackrel{\text{def}}{=} [\hat{A}\hat{B}]$$

onto the completion \mathcal{H}_ω of this pre-HILBERT space⁵⁷ yields a representation π_ω of \mathcal{A} in \mathcal{H}_ω , the so-called **GNS-representation** of \mathcal{A} given by ω . With $\Omega_\omega \stackrel{\text{def}}{=} [\hat{1}]$ we have

$$\omega(\hat{A}) = \langle \Omega_\omega | \pi_\omega(\hat{A}) \Omega_\omega \rangle \quad \forall \hat{A} \in \mathcal{A}$$

and the vector Ω_ω is **cyclic** with respect to $\pi_\omega(\mathcal{A})$; i.e. $\overline{\pi_\omega(\mathcal{A})\Omega_\omega} = \mathcal{H}_\omega$.

Exercise 13 Prove Theorem 1.3.8.⁵⁸

Theorem 1.3.9 *Let ω be a normal state on the VON NEUMANN algebra \mathcal{M} . Then the GNS-representation π_ω is normal, i.e.*

$$\sup_{\mathcal{L}(\mathcal{H})} \left\{ \pi_\omega(\hat{A}_i) : i \in I \right\} = \pi_\omega \left(\sup_{\mathcal{M}} \left\{ \hat{A}_i : i \in I \right\} \right)$$

holds⁵⁹ for every increasing uniformly bounded net $\{\hat{A}_i\}_{i \in I} \subset \mathcal{M}$. Moreover, $\pi_\omega(\mathcal{M})$ is a VON NEUMANN subalgebra of $\mathcal{L}(\mathcal{H}_\omega)$.

Proof: See (Bratteli and Robinson, 1979, Theorem 2.4.24). ■

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⁵⁷For general $*$ -algebras this extension should be skipped; see also (Antoine and Ôta, 1989).

⁵⁸**Hint:** First of all show, using SCHWARZ's inequality (statement (i) of Exercise 7), that $[0]$ is a **left ideal** of \mathcal{A} , i.e. that $[0]$ is a linear subspace of \mathcal{A} and that $\hat{A}[0] \subset [0] \quad \forall \hat{A} \in \mathcal{A}$ (see (Bratteli and Robinson, 1979, Sect. 2.2.3)).

⁵⁹This means, if ω' is a normal state on $\pi_\omega(\mathcal{M})$ then $\hat{\omega}(\hat{A}) \stackrel{\text{def}}{=} \omega'(\pi_\omega(\hat{A}))$ is a normal state on \mathcal{M} .

Definition 1.3.10 Let \mathcal{A} be a C^* -algebra and let π_1, π_2 be representations of \mathcal{A} in \mathcal{H}_1 resp. \mathcal{H}_2 . Then π_2 is said to be **unitarily equivalent** to π_1 iff there is a unitary mapping \hat{U} of \mathcal{H}_1 onto \mathcal{H}_2 fulfilling

$$\hat{U} \pi_1(\hat{A}) = \pi_2(\hat{A}) \hat{U} \quad \forall \hat{A} \in \mathcal{A}.$$

Corollary 1.3.11 Let ω be a state on the C^* -algebra \mathcal{A} with $\hat{1}$, let π be a representation of \mathcal{A} in \mathcal{H} , and let Ω be a cyclic Vector (with respect to $\pi(\mathcal{A})$) fulfilling

$$\omega(\hat{A}) = \langle \Omega | \hat{A} \Omega \rangle \quad \forall \hat{A} \in \mathcal{A}$$

Then, according to Theorem 1.3.8, π is unitarily equivalent to the GNS-representation π_ω .

Exercise 14 Prove Corollary 1.3.11 and show⁶⁰ that – contrary to what BRATTELI and ROBINSON claim (Bratteli and Robinson, 1979, beginning of Section 2.4.4) – equality of the sets of vector states belonging to the representations π_1, π_2 of \mathcal{A} does **not** imply unitary equivalence of π_1 and π_2 , in general.⁶¹

Definition 1.3.12 Let π be a representation in \mathcal{H} of the C^* -algebra \mathcal{A} . Then π is said to be (topologically) **irreducible** iff \mathcal{H} and $\{0\}$ are the only closed subspaces of \mathcal{H} that are mapped into themselves by all $\hat{A} \in \mathcal{A}$.⁶² Otherwise π is said to be **reducible**.

Exercise 15 Let $\mathcal{H}_1, \mathcal{H}_2$ be complex HILBERT spaces and let ω be a pure normal state on the VON NEUMANN subalgebra

$$\mathcal{M} \stackrel{\text{def}}{=} \left\{ \hat{A}_1 \otimes \hat{1} \mid \hat{A}_1 \in \mathcal{L}(\mathcal{H}_1) \right\}$$

of $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Show that the GNS-representation π_ω of \mathcal{M} is irreducible and unitarily equivalent to the representation

$$\pi(\hat{A}_1 \otimes \hat{1}) \stackrel{\text{def}}{=} \hat{A}_1 \quad \forall \hat{A}_1 \in \mathcal{L}(\mathcal{H}_1)$$

of \mathcal{M} in \mathcal{H}_1 .

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⁶⁰Inspect the direct sums $\pi_1 \stackrel{\text{def}}{=} \bigoplus_{t \in \mathbf{R}^1} \pi$ and $\pi_2 \stackrel{\text{def}}{=} \bigoplus_{n=1}^{\infty} \pi$ of one and the same representation π .

⁶¹In case both π_1 and π_2 are irreducible BRATTELI's and ROBINSON's statement is a consequence of Corollary 1.3.11 and Theorem 1.3.13, below.

⁶²That cyclicity does **not** imply irreducibility, in general, can be clearly seen by inspecting the algebra of all functions of the position observable in the SCHRÖDINGER representation (compare Section 1.3.3).

Theorem 1.3.13 *Let ω be a state on the C^* -algebra \mathcal{A} with $\hat{1}$ and let π_ω be the corresponding GNS-representation of \mathcal{A} in \mathcal{H}_ω . Then the following four statements are equivalent:*

1. π_ω is irreducible.
2. ω is pure.
3. Every $\Psi \in \mathcal{H}_\omega \setminus \{0\}$ is cyclic with respect to $\pi_\omega(\mathcal{A})$.
4. $(\pi_\omega(\mathcal{A}))' = \{\alpha\hat{1} : \alpha \in \mathbb{C}\}$.

Proof: See (Bratteli and Robinson, 1979, Prop. 2.3.8 and Theorem 2.3.19.) ■

Exercise 16 Let \mathcal{A} be a C^* -algebra with $\hat{1}$. Prove the following three statements:⁶³

- (i) \mathcal{A} is the linear span of all its unitary⁶⁴ elements \hat{U} .
- (ii) Every representation of \mathcal{A} is unitarily equivalent to a suitable direct sum of either cyclic or trivial representations of \mathcal{A} .
- (iii) A cyclic representation π of \mathcal{A} is unitarily equivalent to a suitable GNS-representation iff it is nontrivial, i.e. iff $\pi(\hat{A}) \neq 0$ for at least one $\hat{A} \in \mathcal{A}$.

Concluding remark: Let \mathcal{A} be a C^* -algebra (with $\hat{1}$). Then one may prove (Bratteli and Robinson, 1979, Lemma 2.3.23) that for every $\hat{A} \in \mathcal{A} \setminus \{0\}$ there is a state ω on \mathcal{A} for which $\omega(\hat{A}) \neq 0$. Hence, if $E_{\mathcal{A}}$ denotes the set of all states on \mathcal{A} ,

$$\pi = \bigoplus_{\omega \in E_{\mathcal{A}}} \pi_\omega$$

is a **faithful** representation of \mathcal{A} . This shows that every C^* -algebra is $*$ -isomorphic to a suitable C^* -subalgebra of $\mathcal{L}(\mathcal{H})$, for suitable \mathcal{H} !

⁶³**Hint for (i):** First show that the formal TAYLOR expansion of $\hat{B} \pm i\sqrt{\hat{1} - \hat{B}^2}$ with respect to the ‘variable’ \hat{B} converges in \mathcal{A} for self-adjoint $\hat{B} \in \mathcal{A}$ with $\|\hat{B}\| < 1$ and the limit is unitary.

Hint for (ii): Use (i) and ZORN’s lemma.

⁶⁴Here, unitarity of \hat{U} means $\hat{U}^* = \hat{U}^{-1}$.

1.3.3 Canonical Quantization

In elementary quantum mechanics of n ‘1-dimensional’ distinguishable particles without inner degrees of freedom and without (further) constraints one uses the state space

$$\mathcal{H}_n \stackrel{\text{def}}{=} L^2(\mathbb{R}^n, dx_1, \dots, dx_n)$$

and the time-zero position operators

$$(\hat{x}_\nu \Psi)(x_1, \dots, x_n) \stackrel{\text{def}}{=} x_\nu \Psi(x_1, \dots, x_n) \quad (1.39)$$

with obvious domains.⁶⁵

Exercise 17 Let \mathcal{M}_{pos} denote the smallest VON NEUMANN subalgebra of $\mathcal{L}(\mathcal{H}_n)$ containing all bounded functions of every \hat{x}_ν . Prove the following four statements:

- (i) \mathcal{M}_{pos} is *maximally abelian*, i.e. \mathcal{M}_{pos} coincides with its commutant⁶⁶

$$\mathcal{M}'_{\text{pos}} = \left\{ \hat{B} \in \mathcal{L}(\mathcal{H}_n) : [\hat{B}, \hat{A}] = 0 \forall \hat{A} \in \mathcal{M}_{\text{pos}} \right\}$$

- (ii) \mathcal{H}_n contains a dense set of vectors which are all cyclic with respect to the identical representation of \mathcal{M}_{pos} .
- (iii) The identical representation of \mathcal{M}_{pos} is reducible.
- (iv) The VON NEUMANN logic of \mathcal{M}_{pos} does not contain any atom.

Translation of particle ν at time zero by a_ν corresponds to a symmetry of the time zero standard logic. The corresponding *-automorphism α_{a_ν} is implemented by the unitary operator $\hat{U}_\nu(-a_\nu)$ defined by

$$[\hat{U}_\nu(-a_\nu)\psi](x_1, \dots, x_n) \stackrel{\text{def}}{=} \psi(x_1, \dots, x_\nu - a_\nu, \dots, x_n), \quad (1.40)$$

i.e.:

$$\alpha_{a_\nu}(\hat{A}) = \hat{U}_\nu(-a_\nu) \hat{A} \hat{U}_\nu(-a_\nu)^{-1}.$$

By Equation (1.39), the operators $\hat{U}_\nu(\tau)$ for fixed ν fulfill the relation

$$\hat{U}_\nu(\tau_1)\hat{U}_\nu(\tau_2) = \hat{U}_\nu(\tau_1 + \tau_2)$$

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⁶⁵Compare Exercise 2, Statement (i), and Equation (1.23).

⁶⁶First show that

$$\begin{aligned} & \int \overline{\phi(x_1, \dots, x_n)} (\hat{T}\psi)(x_1, \dots, x_n) dx_1 \dots dx_n \\ &= \int \overline{\phi(x_1, \dots, x_n)} (\hat{T}\chi_M)(x_1, \dots, x_n) \psi(x_1, \dots, x_n) dx_1 \dots dx_n \end{aligned}$$

holds for $\hat{T} \in \mathcal{M}'_{\text{pos}}$ and $\phi, \psi \in \mathcal{H}_n$ whenever $\text{supp}\phi \subset M$.

and depend continuously on the parameter τ , i.e. they form a **continuous 1-parameter unitary group**. Hence, by STONE's theorem, there exist unique self-adjoint operators $\hat{p}_1, \dots, \hat{p}_n$ with

$$\hat{U}_\nu(\tau) = e^{+\frac{i}{\hbar}\hat{p}_\nu\tau} \quad \text{for } \nu = 1, \dots, n. \quad (1.41)$$

\hat{p}_ν is interpreted as time zero momentum operator for particle ν . If ψ is sufficiently regular we have the TAYLOR expansion

$$\psi(x_1, \dots, x_\nu + \tau, \dots, x_n) = e^{\tau\partial_{x_\nu}}\psi(x_1, \dots, x_n)$$

which, by (1.40) and (1.41), gives

$$(\hat{p}_\nu\psi)(x_1, \dots, x_n) = \frac{\hbar}{i}\partial_{x_\nu}\psi(x_1, \dots, x_n). \quad (1.42)$$

This, finally, yields the **canonical commutation relations**

$$\left. \begin{aligned} [\hat{p}_\nu, \hat{x}_\mu] &= \frac{\hbar}{i}\delta_{\nu\mu}\hat{1} \\ [\hat{x}_\nu, \hat{x}_\mu] &= [\hat{p}_\nu, \hat{p}_\mu] \end{aligned} \right\} \text{ on a suitable domain.} \quad (1.43)$$

Exercise 18 Explain why HEISENBERG's uncertainty relations are not valid for the angular momentum L_3 and its corresponding angular variable $\varphi \in [0, 2\pi)$ of an ordinary quantum mechanical particle (in \mathbb{R}^3) even though their observables obey the canonical commutation relations

$$[\hat{L}_3, \hat{\varphi}] = \frac{\hbar}{i}\hat{1} \quad \text{on some invariant dense domain.}$$

In order to avoid domain problems one replaces the fundamental relations (1.43) by the corresponding ones obeyed by the continuous 1-parameter groups (1.41) and

$$\hat{V}_\mu(s) \stackrel{\text{def}}{=} e^{i\hat{x}_\mu s} \quad (1.44)$$

i.e. by the so-called WEYL **relations**:

$$\begin{aligned} \hat{U}_\nu(\tau)\hat{V}_\mu(s) &= e^{i\tau s\delta_{\nu\mu}}\hat{V}_\mu(s)\hat{U}_\nu(\tau), \\ \hat{U}_\nu(\tau_1)\hat{U}_\nu(\tau_2) &= \hat{U}_\nu(\tau_1 + \tau_2), \quad \hat{V}_\mu(s_1)\hat{V}_\mu(s_2) = \hat{V}_\mu(s_1 + s_2), \\ \hat{U}_\nu(-\tau) &= \hat{U}_\nu(\tau)^* \neq 0 \neq \hat{V}_\mu(-s) = \hat{V}_\mu(s)^*. \end{aligned} \quad (1.45)$$

Remark: (1.45) shows that

$$(p_1, \dots, p_n; x_1, \dots, x_n; t) \longmapsto e^{2\pi i h t} e^{\pi i h \sum_{\nu=1}^n p_\nu x_\nu} \prod_{\mu=1}^n \hat{V}_\mu(2\pi x_\mu) \hat{U}_\mu(h p_\mu)$$

is a representation of the HEISENBERG **group** \mathbf{H}_n , i.e. of \mathbb{R}^{2n+1} with multiplication

$$\stackrel{\text{def}}{=} (p_1, \dots, p_n; x_1, \dots, x_n; t)(p'_1, \dots, p'_n; x'_1, \dots, x'_n; t') \\ \stackrel{\text{def}}{=} (p_1 + p'_1, \dots, p_n + p'_n; x_1 + x'_1, \dots, x_n + x'_n; t + t' + \frac{1}{2} \sum_{\nu=1}^n (p_\nu x'_\nu - p'_\nu x_\nu))$$

(compare (Folland, 1989, Sect. 1.2)).

Theorem 1.3.14 *There is a C^* -algebra \mathcal{A}_n^{B} , unique up to $*$ -isometry, that is generated by elements $\hat{U}_\nu(\tau), \hat{V}_\mu(s)$ ($\nu, \mu \in \{1, \dots, n\}; \tau, s \in \mathbb{R}$) fulfilling the WEYL relations.*

Proof: See (Bratteli and Robinson, 1981, Theorem 5.2.8). ■

The representation of the so-called **CCR-algebra** \mathcal{A}_n^{B} given by the C^* -subalgebra of $\mathcal{L}(\mathcal{H}_n)$ generated by all $\hat{U}_\nu(\tau)$ and $\hat{V}_\mu(s)$ defined above is called the **SCHRÖDINGER representation**. From now on, let us identify \mathcal{A}_n^{B} with its SCHRÖDINGER representation.

Exercise 19 Prove⁶⁷ that the SCHRÖDINGER representation of \mathcal{A}_n^{B} is **irreducible**.

Every state ω on \mathcal{A}_n^{B} is uniquely fixed by⁶⁸ its **generating functional**

$$E_\omega(z_1, \dots, z_n) \stackrel{\text{def}}{=} \omega \left(\hat{W}_1(z_1) \dots \hat{W}_n(z_n) \right), \quad (1.46)$$

$$\text{where: } \hat{W}_\nu(s + i\tau) \stackrel{\text{def}}{=} \hat{U}_\nu\left(\frac{\tau}{2}\right) \hat{V}_\nu(s) \hat{U}_\nu\left(\frac{\tau}{2}\right).$$

If one takes for ω the so-called **FOCK ground state**⁶⁹

$$\omega_{\text{F}}(\hat{A}) \stackrel{\text{def}}{=} \left\langle \Omega_{\text{F}} \left| \hat{A} \Omega_{\text{F}} \right. \right\rangle, \quad \text{where: } \Omega_{\text{F}}(x_1, \dots, x_n) = \pi^{-\frac{n}{4}} e^{-\frac{1}{2}(x_1^2 + \dots + x_n^2)},$$

one gets the so-called **FOCK functional**

$$E_{\text{F}}(s_1, \tau_1; \dots) \stackrel{\text{def}}{=} E_{\omega_{\text{F}}}(s_1 + i\tau_1; \dots),$$

which uniquely characterizes the so-called **FOCK representation**, i.e. the GNS-representation $\pi_{\text{F}} \stackrel{\text{def}}{=} \pi_{\omega_{\text{F}}}$. According to Exercise 19 and Corollary 1.3.11 the FOCK representation is unitary equivalent to the SCHRÖDINGER representation.

Exercise 20 Show that⁷⁰

$$E_{\text{F}}(s_1, \tau_1; \dots; s_n, \tau_n) = \exp \left(-\frac{1}{4}(s_1^2 + \tau_1^2 + \dots + s_n^2 + \tau_n^2) \right).$$

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⁶⁷Recall Statement (i) of Exercise 17 and Theorem 1.3.13.

⁶⁸Recall (1.45) and Exercise 13.

⁶⁹The ground state of n independent harmonic oscillators with mass and circular frequency equal to 1.

⁷⁰Recall (1.40) and note that $\int_{-\infty}^{+\infty} e^{-(z+z_0)^2} dx = \sqrt{\pi}$ for arbitrary complex z_0 .

Definition 1.3.15 A representation π of \mathcal{A}_n^{B} is called **regular** if the $\pi(\hat{U}_\nu(\tau))$ $\pi(\hat{V}_\mu(\tau))$ depend strongly continuously on the parameter τ and coincide with the unit operator for $\tau = 0$.

Obviously, the FOCK representation of \mathcal{A}_n^{B} is regular.

Theorem 1.3.16 (STONE - VON NEUMANN) Let π be a **regular**⁷¹ representation of \mathcal{A}_n^{B} (with **finite** n) in \mathcal{H} . Then π is unitarily equivalent to some direct sum of FOCK representations (resp. SCHRÖDINGER representations) of \mathcal{A}_n^{B} .

Proof: See (Bratteli and Robinson, 1981, Corollary 5.2.15) or (Folland, 1989, pp 35–36). ■

Exercise 21 Using the results of Exercise 20, prove Theorem 1.3.16.⁷²

Let \mathcal{M} denote the VON NEUMANN subalgebra of $\mathcal{L}(\mathcal{H}_{\text{tot}})$ generated by those **non-relativistic** n -particle observables which refer only to the motion of **identifiable** particles with respect to one space dimension but not to inner degrees of freedom. If – as usual – we assume \mathcal{M} to be $*$ -isomorphic to $\mathcal{L}(L^2(\mathbb{R}^{3n}, dx_1, \dots, dx_n))$, interpreting (1.41) and (1.44) in the standard way (compare Exercise 15), we see from Theorem 1.3.16:

The partial states of physically realizable ensembles of the kind described above correspond to **regular** states⁷³ ω on the C^* -algebra \mathcal{A}_n^{B} characterized by the canonical commutation relations in WEYL form. Here the self-adjoint elements of \mathcal{A}_n^{B} may be interpreted as time-zero observables (‘quantum kinematics’) in accord with 1.41, 1.44, and 1.45. Time evolution in the sense of Section 1.1.4 has to be unitary, according to Theorem 1.1.4.

In this sense replacement of the complex numbers (c -numbers) p_ν , q_ν by operators \hat{p}_ν , \hat{q}_ν (q -numbers) fulfilling the commutation Relations (1.43) (as a consequence of (1.45)) is called a **quantization** of the system described above.

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⁷¹For the classification of strongly measurable (not necessarily regular) representations in non-separable HILBERT spaces see (Cavallaro et al., 1998).

⁷²Show, first of all, that

$$\Omega \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} \pi \left(\prod_{\nu=1}^n \left(\int ds e^{-\frac{\epsilon^2}{2} \hat{V}_\nu(s)} \right) \left(\int d\tau e^{-(\epsilon\tau)^2 \hat{U}_\nu(\tau)} \right) \right) \psi$$

is normalized for suitable ψ and that the generating functional E_ω , corresponding to the vector state

$$\omega(\hat{A}) \stackrel{\text{def}}{=} \langle \Omega | \pi(\hat{A}) \Omega \rangle \quad \forall \hat{A} \in \mathcal{A}_n^{\text{B}},$$

coincides with the FOCK functional.

⁷³I.e. π_ω is regular.

For infinitely many degrees of freedom ($n \rightarrow \infty$) the situation is much more complicated:

Let $\{\Psi_j\}_{j \in \mathbb{Z}_+}$ be complete orthonormal system of $\mathcal{H}_1 = L^2(\mathbb{R}^1, dx)$ with $\Psi_0 = \Omega_F(x_1)$ and consider

$$\{(\Psi_{j_1}, \Psi_{j_2}, \dots, \Psi_0, \Psi_0, \dots)\}_{j_1, j_2, \dots \in \mathbb{Z}_+}$$

as a complete orthonormal system of a complex HILBERT space $(\bigotimes_{\nu=1}^{\infty} \mathcal{H}_1)^{\Omega_F^\infty}$, where $\Omega_F^\infty \stackrel{\text{def}}{=} (\Psi_0, \Psi_0, \dots)$, called an *infinite tensor product* of \mathcal{H}_1 with itself. Then, given n , the definition

$$\pi_n(\underbrace{\hat{W}_\nu(z)}_{\in \mathcal{A}_n^B})(\Psi_{j_1}, \dots, \Psi_{j_\nu}, \dots) \stackrel{\text{def}}{=} (\Psi_{j_1}, \dots, \hat{W}_1(z)\Psi_{j_\nu}, \dots),$$

for $\nu = 1, \dots, n$ and complex z (compare (1.46)), fixes a representation π_n of \mathcal{A}_n^B that is unitary equivalent to the corresponding SCHRÖDINGER representation.

The C^* -subalgebra \mathcal{A}_∞^B of $\mathcal{L}\left(\left(\bigotimes_{n=1}^{\infty} \mathcal{H}_1\right)^{\Omega_F^\infty}\right)$ generated by $\bigcup_{n=1}^{\infty} \pi_n(\mathcal{A}_n^B)$ is called the *CCR-algebra*. It is considered to be the algebra of time zero observables of a BOSE system with ‘infinitely many degrees of freedom’ (see also (Yurtsever, 1994) and (Borchers, 1996, Sect. I.2)). Because of

$$\pi_\nu(\mathcal{A}_\nu^B) \subset \pi_{\nu+1}(\mathcal{A}_{\nu+1}^B)$$

the WEYL relations 1.45 hold also for

$$\hat{U}_\nu^\infty(\tau) = \pi_\nu(\hat{U}_\nu(\tau)) \quad \hat{V}_\nu^\infty(s) = \pi_\nu(\hat{V}_\nu(s)),$$

$$\text{where } \hat{U}_\nu(\tau), \hat{V}_\nu(s) \in \mathcal{A}_\nu^B,$$

instead of $\hat{U}_\nu(\tau), \hat{V}_\nu(s)$ for arbitrary $\nu, \mu \in \mathbb{N}$.

The **regular** state ω_F on the *CCR*-algebra with generating functional

$$E_F(z_1, \dots, z_n, 0, 0, \dots) \stackrel{\text{def}}{=} \langle \Omega_F | \hat{W}_1^\infty(z_1), \dots, \hat{W}_n^\infty(z_n) \Omega_F^\infty \rangle$$

(compare (1.46)) is called the FOCK *vacuum*. The corresponding *GNS*-representation $\pi_F = \pi_{\omega_F}$ is called the FOCK *representation* of \mathcal{A}_∞^B .

Exercise 22 Show that the identical representation of \mathcal{A}_∞^B is unitary equivalent to the FOCK representation and **irreducible**.

1.3.4 Spontaneously Broken Symmetries

Let $\hat{U}_1, \hat{U}_2, \dots$ be unitary operators in \mathcal{H}_1 and define

$$\Omega_N^\infty \stackrel{\text{def}}{=} (\hat{U}_1 \Psi_0, \dots, \hat{U}_N \Psi_0, \Psi_0, \Psi_0, \dots) \quad \text{for } N = 1, 2, \dots$$

Then, for finite N , the GNS-representation π_{ω_N} of \mathcal{A}_∞^B for the vector state

$$\omega_N(\hat{A}) \stackrel{\text{def}}{=} \langle \Omega_N^\infty | \hat{A} \Omega_N^\infty \rangle$$

is unitary equivalent to the FOCK representation. In the limit $N \rightarrow \infty$, however, we get a state ω_∞ the GNS-representation π_{ω_∞} of which is **regular**, irreducible, and faithful⁷⁴ but, in general,⁷⁵ **not unitary equivalent** to the FOCK representation.

Exercise 23 Show, by Theorem 1.3.13, that the VON NEUMANN completion of $\pi_{\omega_{N_1}}(\mathcal{A}_\infty^B)$ in $\mathcal{H}_{\omega_{N_1}}$ is isomorphic to the VON NEUMANN completion of $\pi_F(\mathcal{A}_\infty^B)$ in $\mathcal{H}_{\omega_\infty}$. Explain why, nevertheless, $\pi_{\omega_{N_1}}$ may be unitary inequivalent to π_F .

The above construction shows us that Theorem 1.3.16 does not hold for $n = \infty$ but that there is a myriad of – up to now unclassified – physically relevant, regular, irreducible representations of \mathcal{A}_∞^B which are inequivalent to the FOCK representation !

Problem: Which is the VON NEUMANN algebra \mathcal{M} corresponding to \mathcal{A}_∞^B in the sense of Section 1.3.1 and **how** is \mathcal{A}_∞^B embedded into \mathcal{M} ?

As already pointed out, one does not always know the VON NEUMANN algebra \mathcal{M} of the considered partial theory in the sense of Section 1.3.1, but only – up to C^* -algebra isometry – a C^* -subalgebra \mathcal{A} of \mathcal{M} generating \mathcal{M} .

This is the reason for using also C^* -algebras which are not VON NEUMANN algebras, in quantum statistical mechanics and relativistic quantum field theory.

Regarding the physical relevance of states⁷⁶ we then need suitable criteria (such as regularity of states over \mathcal{A}_N^B).

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⁷⁴Limit states of a similar kind play an important role in statistical quantum mechanics (Araki and Woods, 1963) and constructive quantum field theory (Wightman, 1967).

⁷⁵One may prove that

$$(\Phi_1, \Phi_2, \dots) \in \left(\bigotimes_{\nu=1}^{\infty} \mathcal{H}_1 \right)^{\Omega_F^\infty} \iff \sum_{\nu=1}^{\infty} |1 - \langle \Psi_0 | \Phi_\nu \rangle| < \infty.$$

⁷⁶The main problem is to characterize those those partial states on \mathcal{A} which are restrictions of **normal** states on \mathcal{M} .

Definition 1.3.17 A state ω on the C^* -algebra \mathcal{A} is called **normal**⁷⁷ with respect to the representation π on \mathcal{A} in \mathcal{H} , if there is an operator $\hat{T}_\omega \in \mathcal{T}_1(\mathcal{H})$ with

$$\omega(\hat{A}) = \text{Tr} \left(\hat{T}_\omega \pi(\hat{A}) \right) \quad \forall \hat{A} \in \mathcal{A}.$$

The set of all states which are normal with respect to π is called the **folium** \mathcal{S}_π corresponding to π .

Definition 1.3.18 Two representations π_1, π_2 of a C^* -algebra \mathcal{A} are called **quasi equivalent** if their folia coincide, i.e. if $\mathcal{S}_{\pi_1} = \mathcal{S}_{\pi_2}$.

Theorem 1.3.19 Let π_1, π_2 be **non-degenerate** (i.e. $\pi_j(\mathcal{A})\Psi = \{0\} \implies \Psi = 0$) representations of the C^* -algebra \mathcal{A} . Then π_1, π_2 are quasi equivalent if and only if a suitable direct sum of π_1 is unitary equivalent to a suitable sum of π_2 .

Proof: See (Bratteli and Robinson, 1981, Theorem 2.4.26). ■

Definition 1.3.20 A physical symmetry corresponding to the $*$ -automorphism (or $*$ -anti-automorphism) φ of the C^* -algebra \mathcal{A} is said to be **spontaneously broken** by the state ω on \mathcal{A} if the GNS-representations corresponding to ω and $\varphi_*\omega$ are **not** quasi-equivalent, i.e. if $\mathcal{S}_{\pi_\omega} \neq \mathcal{S}_{\pi_{\varphi_*\omega}}$.

Exercise 24 Let \mathcal{A} be a C^* -subalgebra of $\mathcal{L}(\mathcal{H})$ and let \hat{U} be an anti-unitary operator on \mathcal{H} .

(i) Show that

$$\gamma(\hat{A}) \stackrel{\text{def}}{=} \hat{U} \hat{A}^* \hat{U}^* \quad \text{for } \hat{A} \in \mathcal{A}$$

defines a **$*$ -antiautomorphism**, i.e. for all $\hat{A}, \hat{B} \in \mathcal{A}$ and all $z \in \mathbb{C}$:

$$\begin{aligned} (A_1) : \quad & \gamma(\hat{A} + z\hat{B}) = \gamma(\hat{A}) + z\gamma(\hat{B}), \\ (A_2) : \quad & \gamma(\hat{A}\hat{B}) = \gamma(\hat{B})\gamma(\hat{A}), \\ (A_3) : \quad & \gamma(\hat{A}^*) = \gamma(\hat{A})^*. \end{aligned}$$

(ii) Show that γ , as defined above, is the only $*$ -antiautomorphism of \mathcal{A} into $\mathcal{L}(\mathcal{H})$ with

$$\gamma(\hat{P}) = \hat{U} \hat{P} \hat{U}^* \quad \forall \hat{P} \in \mathcal{L}_{\mathcal{H}}.$$

Exercise 25 Show that symmetries may be broken by regular states on \mathcal{A}_∞^B but not by regular states on \mathcal{A}_N^B , $N < \infty$.

Remarks:

- (i) In physically relevant theories (like *QED*) the symmetries corresponding to homogeneous LORENTZ transformations are spontaneously broken (see e.g. (Buchholz, 1986)).
- (ii) For interesting speculations regarding spontaneous breaking of time translation symmetry see (Rieckers, 1985).
- (iii) Spontaneous breaking of gauge symmetries plays a decisive role in the WEINBERG-SALAM theory of electroweak interactions (see e.g. (Mohapatra, 1986) and (Watkins, 1986)).

Exercise 26 Let ω be a state and φ a $*$ -automorphism of the C^* -algebra \mathcal{A} . Show⁷⁸ that the representation π of \mathcal{A} , defined by

$$\pi(\hat{A}) \stackrel{\text{def}}{=} \pi_\omega \left(\varphi^{-1}(\hat{A}) \right) \quad \text{for } \hat{A} \in \mathcal{A},$$

is unitary equivalent to $\pi_{\varphi_*\omega}$ but not necessarily to π_ω .

⁷⁸**Hint:** First, show that

$$\hat{U} \pi_{\varphi_*\omega}(\hat{A}) \Omega_{\varphi_*\omega} \stackrel{\text{def}}{=} \pi_\omega \left(\varphi^{-1}(\hat{A}) \right) \Omega_\omega \quad \text{for } \hat{A} \in \mathcal{A}$$

(consistently!) defines a unitary mapping \hat{U} from $\mathcal{H}_{\varphi_*\omega}$ onto \mathcal{H}_ω .

Chapter 2

Massive Scalar Fields

2.1 Free Neutral Scalar Fields

We are going to describe systems of noninteracting, indistinguishable, relativistic point ‘particles’ on MINKOWSKI space¹ with rest mass $m > 0$ having no internal degrees of freedom and no charge. We use natural units throughout, especially

$$c = \hbar = 1.$$

2.1.1 1-Particle Space

Momentum Space Representation

The *three-momentum* of a classical relativistic point particle is

$$\mathbf{p} = m_{\mathbf{v}} \mathbf{v},$$

where \mathbf{v} is its (three-)velocity and

$$m_{\mathbf{v}} = \frac{m}{\sqrt{1 - |\mathbf{v}|^2}}$$

its *inertial mass* coinciding² – thanks to natural units – with its *energy*

$$p^0 = \omega_{\mathbf{p}} \stackrel{\text{def}}{=} \sqrt{m^2 + |\mathbf{p}|^2} > 0.$$

Therefore,³

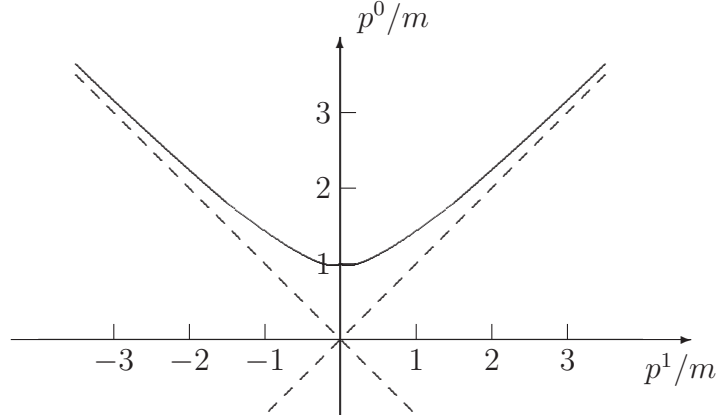
$$\mathbf{v} = \frac{\mathbf{p}}{\omega_{\mathbf{p}}}.$$

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¹For generalization to curved space-time see, e.g. (Verch, 1997) and references given there.

²Just compare the squares of $m_{\mathbf{v}}$ and p^0 .

³This, obviously, is consistent with $\frac{d}{dt}\omega_{\mathbf{p}(t)} = \mathbf{v}(t) \cdot \frac{d}{dt}\mathbf{p}(t)$.

Figure 2.1: 1-particle mass-shell, restricted to the p^0 - p^1 -plane

Moreover, we have

$$\boxed{p \cdot p \stackrel{\text{def}}{=} p^0 p^0 - \mathbf{p} \cdot \mathbf{p} = m^2, \quad p^0 > 0} \quad (2.1)$$

for its *four-momentum*

$$p = (p^0, p^1, p^2, p^3) = (p^0, \mathbf{p}).$$

When changing the inertial system (but not the origin of the resp. system) p has to be transformed by the same LORENTZ matrix as $x = (x^0, \mathbf{x})$:

$$x, p \xrightarrow{\text{change of ref. syst.}} x' = \Lambda x, p' = \Lambda p \quad (2.2)$$

Exercise 27 Let $\{\underline{p}(s)\}_{s \in \mathbb{R}}$ be some (sufficiently well behaved) curve on the **1-particle mass shell** $M_m \stackrel{\text{def}}{=} \{p \in \mathbb{R}^4 : p^0 = \omega_{\mathbf{p}}\}$. Let $(p^0(s), p^1(s), p^2(s), p^3(s))$ resp. $(p'^0(s), p'^1(s), p'^2(s), p'^3(s))$ be the coordinates of $\underline{p}(s)$ in the inertial System L resp. L' , related to each other by a special LORENTZ transformation:

$$p'^0 = \frac{p^0 - u p^1}{\sqrt{1 - u^2}}, \quad p'^1 = \frac{p^1 - u p^0}{\sqrt{1 - u^2}}, \quad p'^2 = p^2, \quad p'^3 = p^3$$

(u fixed). Show that

$$\frac{dp'^1}{ds} = \frac{dp^1}{ds}.$$

Quantum mechanical 1-particle state space:

The pure 1-particle states are given – in the sense of orthodox quantum mechanics – by the vectors of the separable complex HILBERT space

$$\mathcal{H}_0^{(1)} \stackrel{\text{def}}{=} L^2 \left(\mathbb{R}^3, \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}} \right)$$

with inner product

$$\langle \check{f} | \check{g} \rangle \stackrel{\text{def}}{=} \int \overline{\check{f}(\mathbf{p})} \check{g}(\mathbf{p}) \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}} \quad \forall \check{f}, \check{g} \in \mathcal{H}_0^{(1)}. \quad (2.3)$$

Obviously,

$$\left(\hat{U}_0(a, \Lambda) \check{f} \right) (\mathbf{p}) \stackrel{\text{def}}{=} \left(e^{ipa} \check{f} \left(\overrightarrow{\Lambda^{-1}p} \right) \right) \Big|_{p^0=\omega_{\mathbf{p}}} \quad (2.4)$$

defines a representation of \mathcal{P}_+^\uparrow ('restricted' POINCARÉ group), i.e.

$$\hat{U}_0(a_2, \Lambda_2) \hat{U}_0(a_1, \Lambda_1) = \hat{U}_0(\Lambda_2 a_1 + a_2, \Lambda_2 \Lambda_1) \quad \forall (a_2, \Lambda_2), (a_1, \Lambda_1) \in \mathcal{P}_+^\uparrow,$$

which is (strongly) continuous. By Exercise 27 the measure $\frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}$ is LORENTZ invariant. Hence the representation (2.4) is **unitary**, i.e.:

$$\left\langle \hat{U}_0(a, \Lambda) \check{f} \mid \hat{U}_0(a, \Lambda) \check{g} \right\rangle = \langle \check{f} | \check{g} \rangle \quad \forall \check{f}, \check{g} \in \mathcal{H}_0^{(1)}. \quad (2.5)$$

It is to be interpreted as follows:

$$\hat{U}_0(a, \Lambda) \check{f} \text{ corresponds to an ensemble that, with respect to the coordinates } x' \stackrel{\text{def}}{=} \Lambda^{-1}(x - a), \text{ is to be described in exactly the same way as an ensemble corresponding to } \check{f} \text{ is to be described with respect to the coordinates } x. \quad (2.6)$$

According to (2.4), with the projection-valued measure

$$\hat{E}_0(J) \check{f}(\mathbf{p}) \stackrel{\text{def}}{=} \begin{cases} \check{f}(\mathbf{p}) & \text{if } (\omega_{\mathbf{p}}, \mathbf{p}) \in J \\ 0 & \text{otherwise} \end{cases} \quad \text{for BOREL sets } J \subset \mathbb{R}^4 \quad (2.7)$$

we have⁴

$$\hat{U}_0(a) \stackrel{\text{def}}{=} \hat{U}_0(a, \mathbb{1}_4) = \exp \left(i \overline{\hat{P}_0 a} \right) = \int e^{ipa} \hat{E}_0(dp), \quad (2.8)$$

where $\hat{P}_0 \stackrel{\text{def}}{=} \int p \hat{E}_0(dp)$.

⁴As usual, $\overline{\hat{P}_0 a}$ denotes the closure of the (essentially self-adjoint) operator $\hat{P}_0 a$.

\hat{E}_0 is to be interpreted as the spectral measure for 4-momentum, i.e. for $\|\check{f}\| = 1$:

$$\begin{aligned} \langle \check{f} | \hat{E}_0(J) \check{f} \rangle & \stackrel{(2.7)}{=} \int_{(\omega_{\mathbf{p}}, \mathbf{p}) \in J} |\check{f}(\mathbf{p})|^2 \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}} \\ & \stackrel{\text{Interpret.}}{=} \text{probability for: } p \in J \text{ in a state } \hat{=} \check{f} \\ & = \text{probability for: } p \in J \cap M_m \text{ in a state } \hat{=} \check{f}. \end{aligned} \quad (2.9)$$

This is equivalent to interpreting \hat{P}_0 as energy-momentum operator (= observable of 4-momentum):

$$\begin{aligned} \langle \check{f} | \hat{P}_0^\mu \check{f} \rangle & = \int_{p^0 = \omega_{\mathbf{p}}} p^\mu |\check{f}(\mathbf{p})|^2 \frac{d\mathbf{p}}{2p^0} \\ & = \begin{cases} \text{expectation value for the } \mu\text{-component of the} \\ \text{(time-independent) 4-momentum in a state } \hat{=} \check{f} \end{cases} \end{aligned} \quad (2.10)$$

(if $\|\check{f}\| = 1$). This interpretation is also suggested by the relations

$$\hat{U}_0(a, \Lambda)^{-1} \hat{P}_0^\mu \hat{U}_0(a, \Lambda) = \Lambda^\mu{}_\nu \hat{P}_0^\nu \quad (2.11)$$

and

$$\hat{P}_0^0 \geq m \hat{1}, \quad \hat{P}_0 \cdot \hat{P}_0 = m^2 \hat{1}. \quad (2.12)$$

Space-Time Representation

Instead of the $\check{f}(\mathbf{p})$ one may also use the corresponding wave functions⁵

$$f^+(x) \stackrel{\text{def}}{=} (2\pi)^{-3/2} \int_{p^0 = \omega_{\mathbf{p}}} \check{f}(\mathbf{p}) e^{-ipx} \frac{d\mathbf{p}}{2p^0}, \quad (2.13)$$

which uniquely characterize the $\check{f}(\mathbf{p})$ due to

$$\check{f}(\mathbf{p}) = (2\pi)^{-3/2} \int_{p^0 = \omega_{\mathbf{p}}} e^{ipx} i \overleftrightarrow{\partial}_0 f^+(x) dx \quad \forall x^0 \in \mathbb{R}, \quad (2.14)$$

where

$$g(x) \overleftrightarrow{\partial}_0 f(x) \stackrel{\text{def}}{=} g(x) \frac{\partial}{\partial x^0} f(x) - \left(\frac{\partial}{\partial x^0} g(x) \right) f(x). \quad (2.15)$$

Exercise 28 Show, for sufficiently well behaved $\check{f}(\mathbf{p})$, that

$$J_\mu(x) \stackrel{\text{def}}{=} \overline{f^+(x)} i \overleftrightarrow{\partial}_\mu f^+(x)$$

⁵Assume the $\check{f}(\mathbf{p})$ to be sufficiently well behaved, for the moment.

is a conserved LORENTZ vector field⁶ with

$$\int j^0(x) \, d\mathbf{x} = \|\check{f}\|^2 \quad \forall x^0 \in \mathbb{R},$$

but that $j^0(x)$ is not nonnegative,⁷ in general.

The $f^+(x)$ transform according to

$$\check{g}(\mathbf{p}) = \hat{U}_0(a, \Lambda) \check{f}(\mathbf{p}) \xrightarrow{(2.4)} g^+(x) = f^+(\Lambda^{-1}(x - a)) \quad (2.16)$$

and are solutions of the KLEIN-GORDON *equation*

$$(\square + m^2) f^+(x) = 0 \quad (2.17)$$

($\square \stackrel{\text{def}}{=} (\partial_0)^2 - \nabla \cdot \nabla$). The idea is that $|f^+(x)|^2$ describes at least roughly the localization in space-time. There are fundamental obstructions⁸ for defining a position operator in the sense of Sect. 1.2.2 (Hegerfeldt, 1974), (Hegerfeldt and Ruijsenaars, 1980). The most natural definition⁹ would be the one given by NEWTON and WIGNER, according to which – in full analogy to nonrelativistic quantum mechanics – the 3-dimensional FOURIER transform of the time-dependent *momentum amplitude*

$$e^{-i\omega_{\mathbf{p}}x^0} \frac{\check{f}(\mathbf{p})}{\sqrt{2\omega_{\mathbf{p}}}}$$

(compare (2.9)) is interpreted as *position amplitude*; i.e.

$$\int_V |f_{\text{N.W.}}(x)|^2 \, d\mathbf{x} = \text{probability for “}\mathbf{x} \in V \text{ at time } x^0 \text{ in a state } \hat{=} \check{f}” \quad (2.18)$$

(if $\|f\| = 1$), where¹⁰

$$\begin{aligned} \check{f}_{\text{N.W.}}(\mathbf{p}) &= \frac{\check{f}(\mathbf{p})}{\sqrt{2\omega_{\mathbf{p}}}}, \\ f_{\text{N.W.}}(x) &\stackrel{(2.13)}{=} (2\pi)^{-3/2} \int_{p^0=\omega_{\mathbf{p}}} \check{f}_{\text{N.W.}}(\mathbf{p}) e^{-ipx} \, d\mathbf{p}. \end{aligned} \quad (2.19)$$

Here one easily realizes the following problem:¹¹

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⁶I.e.: $\partial_\mu j^\mu = 0$, $\overline{f^+(\Lambda^{-1}(x - a))} i \overleftrightarrow{\partial}_\mu f^+(\Lambda^{-1}(x - a)) = \Lambda^\mu{}_\nu j^\nu(\Lambda^{-1}(x - a))$.

⁷Check $j^0(0)$ for

$$\check{f}(\mathbf{p}) \sim \omega_{\mathbf{p}} \left(\delta_\epsilon(\mathbf{p} - \mathbf{p}_1) - \frac{1}{2} \delta_\epsilon(\mathbf{p} - \mathbf{p}_2) \right)$$

where δ_ϵ is sufficiently close to the delta function and $\mathbf{p}_1, \mathbf{p}_2$ are fixed momenta with $|\mathbf{p}_1| < |\mathbf{p}_2|$.

⁸From the relativistic point of view this is quite satisfactory (see (Crewther, 1995, Sect. 1)).

⁹See (Wightman, 1962) for a very detailed discussion.

¹⁰Note that $\|\check{f}\| = \int |f_{\text{N.W.}}(x)|^2 \, d\mathbf{x}$.

¹¹See Exercise 29, below. A consistent definition of strict localization for relativistic quantum field theory was given in (Knight, 1961) – not on the 1-particle level, of course.

Even though – in the relativistic theory – the velocity is bounded by $|\mathbf{v}| \leq 1$, there can exist at most one instant of time at which the ‘particle’ is localized within a bounded space region in the sense of NEWTON and WIGNER!

As usual, let us denote, for $n \in \mathbb{N}$, by $\mathcal{S}(\mathbb{R}^n)$ the linear topological space of all complex-valued C^∞ functions $\varphi(\xi_1, \dots, \xi_n)$ on \mathbb{R}^n for which all the norms

$$\|\varphi\|_N \stackrel{\text{def}}{=} \sup_{(\xi_1, \dots, \xi_n) \in \mathbb{R}^n} \left| 1 + \sum_{\nu=1}^n \xi_\nu^2 \right|^N \sup_{\substack{(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n \\ \alpha_1 + \dots + \alpha_n < N}} \left| \left(\frac{\partial}{\partial \xi_1} \right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \xi_n} \right)^{\alpha_n} \varphi(\xi_1, \dots, \xi_n) \right|$$

($N \in \mathbb{N}$) are finite; the sets

$$U_{N, \epsilon} \stackrel{\text{def}}{=} \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \|\varphi\|_N < \epsilon\}$$

($N \in \mathbb{N}$, $\epsilon > 0$) forming a basis of open neighborhoods of 0, by definition.

A function $f^+(x)$ is called a **positive frequency smooth KLEIN-GORDON solution** iff it is of the form (2.13) with $\check{f} \in \mathcal{S}(\mathbb{R}^3)$. The function $f^-(x)$ is called a **negative frequency smooth KLEIN-GORDON solution** iff it is the complex conjugate of a positive frequency KLEIN-GORDON solution: $f^-(x) = \overline{f^+(x)}$. Finally, the function $f(x)$ is called a **smooth KLEIN-GORDON solution** iff it is of the form

$$f(x) = f^+(x) + f^-(x)$$

with $f^+(x)$ resp. $f^-(x)$ a positive resp. negative frequency smooth KLEIN-GORDON solution.¹²

Exercise 29 Let f^+ be a positive frequency smooth KLEIN-GORDON solution. Using the easy part of the PALEY-WIENER theorem (see, e.g., (Gårding and Lions, 1959, Theorem 7.1.5)), saying that the FOURIER transform of a (generalized) function with compact support is an entire analytic function, prove the following statements:

- (i) There is at most one instant of time x^0 for which $f^+(x)$, considered as a function of \mathbf{x} , vanishes outside some bounded subset of \mathbb{R}^3 .
- (ii) There is no instant of time x^0 for which both $f^+(x)$ and $\frac{\partial}{\partial x^0} f^+(x)$ vanish outside some bounded subset of \mathbb{R}^3 .

As mentioned above, in spite of the obstructions for defining a fully satisfactory position operator, the transformation rule (2.16) suggests that $|f^+(x)|^2$ describes at least roughly the localization in space-time for a particle with momentum amplitude $\check{f}(\mathbf{p})$. This expectation is confirmed by the following Lemma.¹³

————— Draft, March 26, 2009 —————

¹²Note that this decomposition of $f(x)$ is unique.

¹³A weaker version of Lemma 2.1.1 was originally proved in (Ruelle, 1962).

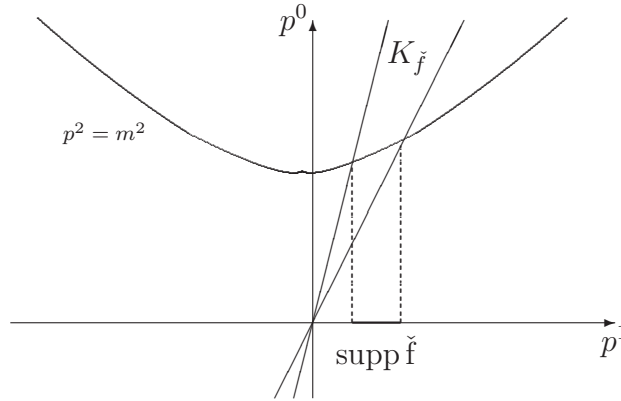


Figure 2.2: Velocity cone $K_{\check{f}}$ (= asymptotic localization region for $f(x)$), restricted to the p^0 - p^1 -plane

Lemma 2.1.1 (Ruelle) *Let $f^+(x)$ be a positive frequency smooth KLEIN-GORDON solution. Then for every $N \in \mathbb{Z}_+$ there is a constant C for which*

$$\begin{aligned} & \| (t, \mathbf{v}t) \|^N |f^+(t - x^0, \mathbf{v}t - \mathbf{x})| \\ & \leq (1 + \|x\|)^N C \quad \forall x \in \mathbb{R}^4, t \in \mathbb{R}^1, \mathbf{v} \in \mathbb{R}^3 \setminus \left\{ \frac{\mathbf{p}}{\omega_{\mathbf{p}}} : \mathbf{p} \in \text{supp } \check{f} \right\}. \end{aligned}$$

Proof: See (Lücke, 1974b, Appendix 2). ■

Exercise 30 Proof Lemma 2.1.1 for the special case

$$\mathbf{v} = (v, 0, 0), v \notin \left\{ \frac{p^1}{\omega_{\mathbf{p}}} : \mathbf{p} \in \text{supp } \check{f} \right\}, \quad x = 0$$

by substitution of variables

$$p^1 \longrightarrow \xi \stackrel{\text{def}}{=} \omega_{\mathbf{p}} - \mathbf{p} \cdot \mathbf{v}$$

and N -fold partial integration¹⁴ with respect to ξ .

2.1.2 FOCK SPACE

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¹⁴Note that $dp^1 = \frac{\omega_{\mathbf{p}(\xi)}}{p^1(\xi) - v\omega_{\mathbf{p}(\xi)}} d\xi$ and rewrite differentiation with respect to ξ as differentiation with respect to p^1 after partial integration.

Free n -Particle System (Momentum Representation)

As in nonrelativistic quantum mechanics, n -particle states are described by functions of n -times as many variables as in the 1-particle case. Since the particles cannot be distinguished and have spin 0, we require these functions to be symmetric with respect to exchange of 3-momenta:¹⁵

$$\begin{aligned} \mathcal{H}_0^{(n)} = & \left\{ \check{f}_n(\mathbf{p}_1, \dots, \mathbf{p}_n) \in L^2\left(\mathbb{R}^{3n}, \frac{d\mathbf{p}_1 \cdots d\mathbf{p}_n}{2\omega_{\mathbf{p}_1} \cdots 2\omega_{\mathbf{p}_n}}\right) : \right. \\ & \left. \check{f}_n(\mathbf{p}_{\pi 1}, \dots, \mathbf{p}_{\pi n}) = \check{f}_n(\mathbf{p}_1, \dots, \mathbf{p}_n) \quad \forall \pi \in S_n \right\}, \\ \langle \check{f}_n | \check{g}_n \rangle \stackrel{\text{def}}{=} & \int \overline{\check{f}_n(\mathbf{p}_1, \dots, \mathbf{p}_n)} \check{g}_n(\mathbf{p}_1, \dots, \mathbf{p}_n) \frac{d\mathbf{p}_1 \cdots d\mathbf{p}_n}{2\omega_{\mathbf{p}_1} \cdots 2\omega_{\mathbf{p}_n}}. \end{aligned} \quad (2.20)$$

Again, the corresponding representation

$$\left(\hat{U}_0(a, \Lambda) \check{f}_n \right)(\mathbf{p}_1, \dots, \mathbf{p}_n) \stackrel{\text{def}}{=} \left(e^{i(p_1 + \dots + p_n)a} \check{f}_n \left(\overrightarrow{\Lambda^{-1} p_1}, \dots, \overrightarrow{\Lambda^{-1} p_n} \right) \right) \Big|_{p_j^0 = \omega_{\mathbf{p}_j}} \quad (2.21)$$

of \mathcal{P}_+^\dagger fulfills (2.5) (unitarity), (2.8) (4-dim. spectral representation), and (2.11) (transformation behaviour of \hat{P}_0).

Exercise 31 Determine the spectral measure \hat{E}_0 of \hat{P}_0 on \mathbb{R}^4 (generalization of (2.7)).

Exercise 32 Show that for every $n \in \mathbb{N}$ and every function w on $\{0, 1\}^n$ the equation

$$\sum_{\nu=1}^n \nu \sum_{(b_1, \dots, b_n) \in M_\nu} w(b_1, \dots, b_n) = \sum_{\mu=1}^n \sum_{\substack{(b_1, \dots, b_n) \in \{0, 1\}^n \\ b_\mu = 1}} w(b_1, \dots, b_n)$$

holds where

$$M_\nu \stackrel{\text{def}}{=} \left\{ (b_1, \dots, b_n) \in \{0, 1\}^n : \sum_{\mu=1}^n b_\mu = \nu \right\} \quad \text{for } \nu = 1, \dots, n.$$

If the particles could be distinguished we had

$$\int_{\mathbf{p}_\nu \in B_3} |\check{f}_n(\mathbf{p}_1, \dots, \mathbf{p}_n)|^2 \frac{d\mathbf{p}_1 \cdots d\mathbf{p}_n}{2\omega_{\mathbf{p}_1} \cdots 2\omega_{\mathbf{p}_n}} = \left\{ \begin{array}{l} \text{probability for particle } \nu \text{ having a} \\ \text{3-momentum } \mathbf{p}_\nu \in B_3 \text{ in a state } \hat{=} \check{f}_n \end{array} \right.$$

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¹⁵As usual, we denote by S_n the group of all permutation of n elements. We do not consider the – much more complicated – possibility of para-BOSE statistics (see e.g. (Ohmuki and Kamefuchi, 1982) and references given there; especially (Stolt and Taylor, 1970), (Hartle and Taylor, 1969)).

under obvious conditions. Therefore, according to Exercise 32, (2.9) becomes

$$\int_{B_{3n}} \|\hat{a}_0(\mathbf{p})\check{f}_n\|^2 \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}} = \begin{cases} \text{expectation value for the} \\ \text{number of particles with } \mathbf{p} \in B_{3n} \end{cases} \quad (2.22)$$

for normalized $\check{f}_n \in \mathcal{S}(\mathbb{R}^{3n}) \subset \mathcal{H}_0^{(n)}$, $n > 0$, if $\hat{a}_0(\mathbf{p})$ denotes the linear mapping

$$(\hat{a}_0(\mathbf{p})\check{f}_n) \underbrace{(\mathbf{p}_1, \dots, \mathbf{p}_{n-1})}_{\text{absent for } n \leq 1} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{for } n = 0 \\ \sqrt{n}\check{f}_n(\mathbf{p}, \underbrace{\mathbf{p}_1, \dots, \mathbf{p}_{n-1}}_{\text{absent for } n=1}) \in \mathcal{H}_0^{(n-1)} & \text{for } n > 0 \end{cases} \quad (2.23)$$

from $\mathcal{S}(\mathbb{R}^{3n}) \subset \mathcal{H}_0^{(n)}$ into $\mathcal{S}(\mathbb{R}^{3(n-1)}) \subset \mathcal{H}_0^{(n-1)}$; where

$$\mathcal{S}(\mathbb{R}^0) \stackrel{\text{def}}{=} \mathcal{H}_0^{(0)} \stackrel{\text{def}}{=} \mathbb{C}, \quad \langle \check{f}_0 | \check{g}_0 \rangle \stackrel{\text{def}}{=} \overline{\check{f}_0} \check{g}_0.$$

(2.10) becomes

$$\begin{aligned} \langle \check{f}_n | \hat{P}_0^\mu \check{f}_n \rangle &= \int_{p^0 = \omega_{\mathbf{p}}} p^\mu \|\hat{a}_0(\mathbf{p})\check{f}_n\|^2 \frac{d\mathbf{p}}{2p^0} \\ &= \int_{p_j^0 = \omega_{\mathbf{p}_j}} (p_1^\mu + \dots + p_n^\mu) |\check{f}_n(\mathbf{p}_1, \dots, \mathbf{p}_n)|^2 \frac{d\mathbf{p}_1 \cdots d\mathbf{p}_n}{2p_1^0 \cdots 2p_n^0} \quad (2.24) \\ &= \begin{cases} \text{expectation value for the } \mu\text{-component of the} \\ \text{total 4-momentum in a state } \hat{=} \check{f}_n \end{cases} \end{aligned}$$

for normalized $\check{f}_n \in \mathcal{S}(\mathbb{R}^{3n}) \cap \mathcal{H}_0^{(n)}$, but instead of (2.12) we have

$$\hat{P}_0^0 \geq nm\hat{1}, \quad \hat{P}_0 \cdot \hat{P}_0 \geq (nm)^2\hat{1}. \quad (2.25)$$

Total State Space

If one does not want – or even cannot – fix the particle number, it is convenient to identify \mathcal{H}_0 with the so-called **FOCK-space** :

$$\mathcal{H}_0 = \bigoplus_{n=0}^{\infty} \mathcal{H}_0^{(n)}.$$

Here the elements of \mathcal{H}_0 are sequences

$$\underline{\check{f}} \stackrel{\text{def}}{=} \{\check{f}_0, \check{f}_1, \check{f}_2, \dots\}$$

with $\check{f}_n \in \mathcal{H}_0^{(n)}$ for $n = 0, 1, \dots$ and $\|\underline{\check{f}}\| < \infty$, where¹⁶

$$\langle \underline{\check{f}} | \underline{\check{g}} \rangle \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \langle \check{f}_n | \check{g}_n \rangle.$$

¹⁶To be absolutely precise, one should use different symbols for the various inner products.

The corresponding unitary representation $\hat{U}_0(a, \Lambda)$ of \mathcal{P}_+^\dagger in \mathcal{H}_0 is given by¹⁷

$$\hat{U}_0(a, \Lambda)\underline{\check{f}} \stackrel{\text{def}}{=} \left\{ \check{f}_0, \hat{U}_0(a, \Lambda)\check{f}_1, \hat{U}_0(a, \Lambda)\check{f}_2, \dots \right\}.$$

Again, (2.8) and

$$\hat{P}_0^0 \geq 0, \quad \hat{P}_0 \cdot \hat{P}_0 \geq 0.$$

hold, while (2.22) becomes

$$\int_{\mathcal{G}} \|\hat{a}_0(\mathbf{p})\underline{\check{f}}\|^2 \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}} = \left\{ \begin{array}{l} \text{expectation value for the} \\ \text{number of particles with } \mathbf{p} \in \mathcal{G} \end{array} \right. \quad (2.26)$$

for **normalized** $\underline{\check{f}} \in D_0$. Here, the domain D_0 is defined by

$$D_0 \stackrel{\text{def}}{=} \left\{ \underline{\check{f}} \in \mathcal{H}_0 : \check{f}_n \in \mathcal{S}(\mathbb{R}^{3n}) \quad \forall n, \quad \check{f}_n = 0 \quad \forall n > n_0(\underline{\check{f}}) \right\} \quad (2.27)$$

and the **annihilation operator** (field) $\hat{a}_0(\mathbf{p})$ by

$$\hat{a}_0(\mathbf{p})\underline{\check{f}} \stackrel{\text{def}}{=} \left\{ \hat{a}_0(\mathbf{p})\check{f}_1, \hat{a}_0(\mathbf{p})\check{f}_2, \dots \right\} \quad \forall \underline{\check{f}} \in D_0. \quad (2.28)$$

As a consequence of (2.23) and (2.4) we thus have

$$\begin{aligned} \hat{U}_0(a, \Lambda)^{-1}\hat{a}_0(\mathbf{p})\hat{U}_0(a, \Lambda) &= e^{+ipa}\hat{a}_0\left(\overrightarrow{\Lambda^{-1}p}\right) \Big|_{p^0=\omega_{\mathbf{p}}}, \\ \hat{U}_0(a, \Lambda)\hat{a}_0(\mathbf{p})\hat{U}_0(a, \Lambda)^{-1} &= e^{-i\Lambda pa}\hat{a}_0\left(\overrightarrow{\Lambda p}\right) \Big|_{p^0=\omega_{\mathbf{p}}}. \end{aligned} \quad (2.29)$$

According to (2.28), $\hat{a}_0(\mathbf{p})$ annihilates the so-called **vacuum vector** $\Omega_0 \stackrel{\text{def}}{=} \{1, 0, \dots\}$ just as the energy-momentum operator does:

$$\hat{a}_0(\mathbf{p})\Omega_0 = \left(\hat{1} - \hat{U}_0(a, \Lambda)\right)\Omega_0 = \hat{P}_0^\mu\Omega_0 = 0. \quad (2.30)$$

2.1.3 The Free Field

Creation Operators in Momentum Space

From (2.24)/(2.28) we conclude **formally**:

$$\hat{P}_0^\mu = \int_{p^0=\omega_{\mathbf{p}}} \hat{a}_0(\mathbf{p})^* p^\mu \hat{a}_0(\mathbf{p}) \frac{d\mathbf{p}}{2p^0}. \quad (2.31)$$

¹⁷Similarly, to be precise, different symbols should be used for the various subrepresentations.

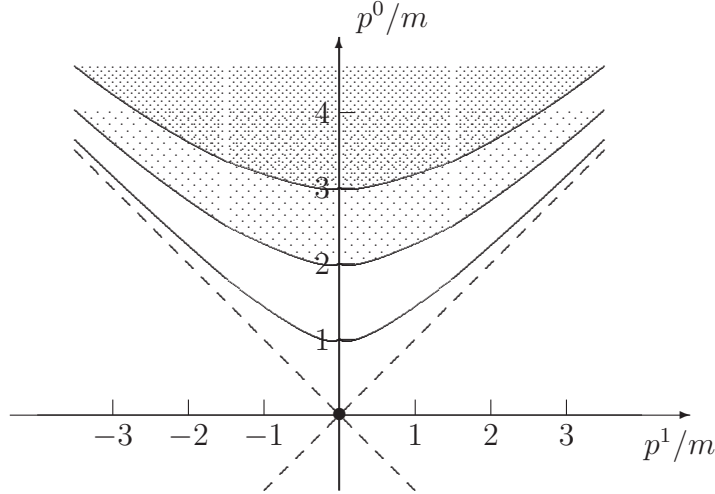


Figure 2.3: Spectrum of \hat{P}_0 (= support of \hat{E}_0), restricted to the p^0 - p^1 -plane

Unfortunately, however, the adjoint $\hat{a}_0(\mathbf{p})^*$ of $\hat{a}_0(\mathbf{p})$ does **not** exist for fixed \mathbf{p} :
The restriction to D_0 of the adjoint $\hat{a}_0(\check{\chi})^*$ of

$$\hat{a}_0(\check{\chi}) \stackrel{\text{def}}{=} \int \hat{a}_0(\mathbf{p}) \check{\chi}(\mathbf{p}) \, d\mathbf{p} \quad \text{for } \check{\chi} \in \mathcal{S}(\mathbb{R}^3)$$

is given by¹⁸

$$\begin{aligned} (\hat{a}_0(\check{\chi})^* \underline{f})_0 &= 0, \\ (\hat{a}_0(\check{\chi})^* \underline{f})_1(\mathbf{p}_1) &= 2\omega_{\mathbf{p}_1} \overline{\check{\chi}(\mathbf{p}_1)} f_0, \\ (\hat{a}_0(\check{\chi})^* \underline{f})_{n+1}(\mathbf{p}_1, \dots, \mathbf{p}_{n+1}) &= \frac{1}{\sqrt{n+1}} \sum_{\nu=1}^{n+1} 2\omega_{\mathbf{p}_\nu} \overline{\check{\chi}(\mathbf{p}_\nu)} \check{f}_n(\mathbf{p}_1, \dots, \mathbf{p}_\nu, \dots, \mathbf{p}_{n+1}), \end{aligned}$$

for $\underline{f} \in D_0$; hence, formally, by

$$(\hat{a}_0(\mathbf{p})^* \underline{f})_{n+1}(\mathbf{p}_1, \dots, \mathbf{p}_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{\nu=1}^{n+1} 2\omega_{\mathbf{p}} \delta(\mathbf{p} - \mathbf{p}_\nu) \check{f}_n(\mathbf{p}_1, \dots, \mathbf{p}_\nu, \dots, \mathbf{p}_{n+1}) \quad (2.32)$$

for $\underline{f} \in D_0$. This means,

$$\hat{a}_0^*(\check{\chi}) \stackrel{\text{def}}{=} \int \hat{a}_0^*(\mathbf{p}) \check{\chi}(\mathbf{p}) \, d\mathbf{p} \stackrel{\text{def}}{=} \left(\int \hat{a}_0(\mathbf{p}) \overline{\check{\chi}(\mathbf{p})} \, d\mathbf{p} \right)^* \wedge D_0 \quad (2.33)$$

creates a particle with momentum space wave function $2\omega_{\mathbf{p}} \check{\chi}(\mathbf{p})$. As a simple consequence of (2.23) we get the canonical commutation relations

$$\boxed{\begin{aligned} [\hat{a}_0(\mathbf{p}), \hat{a}_0(\mathbf{p}')]_- &= [\hat{a}_0^*(\mathbf{p}), \hat{a}_0^*(\mathbf{p}')]_- = 0, \\ [\hat{a}_0(\mathbf{p}), \hat{a}_0^*(\mathbf{p}')]_- &= 2\omega_{\mathbf{p}} \delta(\mathbf{p} - \mathbf{p}'). \end{aligned}} \quad (2.34)$$

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¹⁸We write \mathbf{p}_λ when \mathbf{p}_ν has to be skipped.

The precise meaning of (2.34) is¹⁹

$$\left. \begin{aligned} [\hat{A}_{\check{f}}, \hat{A}_{\check{g}}]_- &= [\hat{A}_{\check{f}}^*, \hat{A}_{\check{g}}^*]_- = 0 \\ [\hat{A}_{\check{f}}, \hat{A}_{\check{g}}^*]_- &= \langle \check{f} | \check{g} \rangle_{\mathcal{H}_0^{(1)}} \hat{1} \end{aligned} \right\} \quad \forall \check{f}, \check{g} \in \mathcal{S}(\mathbb{R}^3)$$

on D_0 , where

$$\hat{A}_{\check{f}} \stackrel{\text{def}}{=} \int \hat{a}_0(\mathbf{p}) \frac{\overline{\check{f}(\mathbf{p})}}{2\omega_{\mathbf{p}}} d\mathbf{p}, \quad \text{hence} \quad \hat{A}_{\check{f}}^* \wedge D_0 = \int \hat{a}_0^*(\mathbf{p}) \frac{\check{f}(\mathbf{p})}{2\omega_{\mathbf{p}}} d\mathbf{p}.$$

The present representation of the canonical commutation relations has the **FOCK property**

$$\hat{A}_{\check{f}} \Omega = 0 \quad \forall \check{f} \in \mathcal{S}(\mathbb{R}^3).$$

Moreover, according to (2.29) and (2.33), we have

$$\begin{aligned} \hat{U}_0(a, \Lambda)^{-1} \hat{a}_0^*(\mathbf{p}) \hat{U}_0(a, \Lambda) &= e^{-ipa} \hat{a}_0^* \left(\overrightarrow{\Lambda^{-1}p} \right) \Big|_{p^0=\omega_{\mathbf{p}}}, \\ \hat{U}_0(a, \Lambda) \hat{a}_0^*(\mathbf{p}) \hat{U}_0(a, \Lambda)^{-1} &= e^{+i\Lambda pa} \hat{a}_0^* \left(\overrightarrow{\Lambda p} \right) \Big|_{p^0=\omega_{\mathbf{p}}}. \end{aligned} \quad (2.35)$$

Expressions like

$$\langle \Psi_1 | \hat{a}_0^*(\mathbf{p}) \Psi_2 \rangle, \quad \Psi_1 \in \mathcal{H}, \quad \Psi_2 \in D_0,$$

being well-defined only when *smearred* by some test function $\check{\chi}$,

$$\int \langle \Psi_1 | \hat{a}_0^*(\mathbf{p}) \Psi_2 \rangle \check{\chi}(\mathbf{p}) d\mathbf{p} \stackrel{\text{def}}{=} \langle \Psi_1 | \hat{a}_0^*(\check{\chi}) \Psi_2 \rangle,$$

($\check{\chi} \in \mathcal{S}(\mathbb{R}^3)$, here), are called **generalized functions** (if linearly and continuously depending on the test function from a suitable topological *test space*). This name indicates that many operations, defined for ordinary functions, may be generalized to these functionals:

Let \hat{K}, \hat{K}' be ‘sufficiently well-behaved’ linear operators fulfilling

$$\int \left(\hat{K} \check{\chi}_1 \right) (\mathbf{p}) \check{\chi}_2(\mathbf{p}) d\mathbf{p} = \int \check{\chi}_1(\mathbf{p}) \hat{K}' \check{\chi}_2(\mathbf{p}) d\mathbf{p} \quad (2.36)$$

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¹⁹Note that, if $\{\check{f}_\nu\}_{\nu \in \mathbb{N}}$ is a complete orthonormal system in $\mathcal{H}_0^{(1)}$, the operators

$$\hat{p}_\nu \stackrel{\text{def}}{=} \frac{\hat{A}_{\check{f}_\nu}^* \wedge D_0 + \hat{A}_{\check{f}_\nu}}{\sqrt{2}}, \quad \hat{x}_\nu \stackrel{\text{def}}{=} \frac{\hat{A}_{\check{f}_\nu}^* \wedge D_0 - \hat{A}_{\check{f}_\nu}}{i\sqrt{2}}$$

fulfill the relations (1.43) with $\hbar = 1$ and that

$$\hat{P}_0 \wedge D_0 = \sum_{\nu=1}^{\infty} \langle f_\nu | \hat{P}_0 f_\nu \rangle \hat{A}_{\check{f}_\nu}^* \hat{A}_{\check{f}_\nu}.$$

for all test functions $\check{\chi}_1, \check{\chi}_2$. then it is natural to define, e.g.,

$$\int \left(\hat{K}F \right) (\mathbf{p}) \check{\chi}(\mathbf{p}) \, d\mathbf{p} \stackrel{\text{def}}{=} \int F(\mathbf{p}) \hat{K}'\check{\chi}(\mathbf{p}) \, d\mathbf{p} \quad \forall \check{\chi} \in \mathcal{S}(\mathbb{R}^3) \quad (2.37)$$

for continuous linear functionals F on $\mathcal{S}(\mathbb{R}^3)$. Correspondingly, we then define

$$\int \left(\hat{K}\hat{a}_0^* \right) (\mathbf{p}) \check{\chi}(\mathbf{p}) \, d\mathbf{p} = \int \hat{a}_0^*(\mathbf{p}) \hat{K}'\check{\chi}(\mathbf{p}) \, d\mathbf{p} \quad (2.38)$$

for test functions $\check{\chi}$. Generalizations of these prescriptions are obvious and will be used without special explanation.

Exercise 33 Determine the following operations on $\hat{a}_0^*(\mathbf{p})$:

- (i) partial differentiation
- (ii) multiplication by suitable functions
- (iii) transformation of variables (e.g. POINCARÉ transformations)
- (iv) FOURIER transformation

Field Operators in MINKOWSKI Space

Similarly to the wave functions $f(x)$ (see (2.13)) one defines the *field operators*²⁰

$$\hat{\Phi}_0^+(x) \stackrel{\text{def}}{=} (2\pi)^{-3/2} \int_{p^0=\omega_{\mathbf{p}}} \hat{a}_0(\mathbf{p}) e^{-ipx} \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}, \quad (2.39)$$

(*positive frequency part* or *creation part*) and

$$\hat{\Phi}_0^-(x) \stackrel{\text{def}}{=} \left(\hat{\Phi}_0^+(x) \right)^* \stackrel{\text{def}}{=} (2\pi)^{-3/2} \int_{p^0=-\omega_{\mathbf{p}}} \hat{a}_0^*(-\mathbf{p}) e^{-ipx} \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}, \quad (2.40)$$

(*negative frequency part* or *annihilation part*) on D_0 , which are both solutions of the KLEIN-GORDON equation²¹

$$(\square + m^2) \hat{\Phi}_0^\pm(x) = 0 \quad (2.41)$$

(in the sense of generalized functions) and, thanks to (2.29), transform according to

$$\hat{U}_0(a, \Lambda) \hat{\Phi}_0^\pm(x) \hat{U}_0(a, \Lambda)^{-1} = \hat{\Phi}_0^\pm(\Lambda x + a). \quad (2.42)$$

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²⁰The exact definition is by multiplication with $\exp(-i\omega_{\mathbf{p}}x^0)/2\omega_{\mathbf{p}}$ and subsequent 3-FOURIER transform w.r.t. \mathbf{p} , both in the sense of generalized functions.

²¹Note that, for $\check{f} \in D_0$, $\langle \check{f} | \hat{\Phi}_0^+(x) \rangle$ resp. $\langle \check{f} | \hat{\Phi}_0^-(x) \rangle$ is a positive resp. negative frequency smooth Klein Gordon solution.

It is convenient to add both parts to get a hermitian field operator

$$\hat{\Phi}_0(x) \stackrel{\text{def}}{=} \hat{\Phi}_0^+(x) + \hat{\Phi}_0^-(x). \quad (2.43)$$

Note that it is sufficient to smear $\hat{\Phi}_0(x)$ in the space variables, i.e.

$$\hat{\Phi}_\psi(x^0) \stackrel{\text{def}}{=} \int \hat{\Phi}_0(x) \psi(\mathbf{x}) \, d\mathbf{x}$$

is well defined for $x^0 \in \mathbb{R}$ and $\psi \in \mathcal{S}(\mathbb{R}^3)$.

Exercise 34 Show for arbitrary $x^0 \in \mathbb{R}$ and $\psi \in \mathcal{S}(\mathbb{R}^3)$:

- (i) $\psi = \bar{\psi} \implies \left(\hat{\Phi}_\psi(x^0)\right)^* \wedge D_0 = \hat{\Phi}_\psi(x^0)$.
- (ii) Every $\check{f} \in D_0$ is an entire analytic vector for $\hat{\Phi}_\psi(x^0)$, i.e.:

$$\sum_{\nu=0}^{\infty} \frac{1}{\nu!} \left\| \left(\lambda \hat{\Phi}_\psi(x^0)\right)^\nu \check{f} \right\| < \infty \quad \forall \lambda > 0.$$
- (iii) $\hat{\Phi}_\psi(x^0)D_0 \subset D_0$.

According to Exercise 34 and a well-known theorem by Nelson (Reed und Simon, 1972, Sect. X.4) $\hat{\Phi}_\psi(x^0)$ has a unique self-adjoint extension if ψ is real-valued. By (2.42), $\hat{\Phi}(x)$ transforms like the observable of the field strength of some LORENTZ invariant scalar field:

$$\hat{U}_0(a, \Lambda) \hat{\Phi}_0(x) \hat{U}_0(a, \Lambda)^{-1} \stackrel{(2.42)}{=} \hat{\Phi}_0(\Lambda x + a).$$

This suggests the interpretation

$$\hat{\Phi}_\psi(x^0) = \begin{cases} \text{observable of the mean value } \int \Phi_0(x) \psi(\mathbf{x}) \, d\mathbf{x} \\ \text{of the classical field}^{22} \Phi_0(x) \text{ at time zero.} \end{cases} \quad (2.44)$$

Since, for real-valued ψ , $\hat{\Phi}_\psi(x_0)$ should correspond to some measurement performable within $\text{supp } \psi$ at time x^0 the condition of **local commutativity**²³

$$\left[\hat{\Phi}_0(x), \hat{\Phi}_0(y) \right]_- = 0 \quad \text{for } x \times y. \quad (2.45)$$

(also called **microcausality** for observable fields) should be fulfilled (compare Footnote 29). Indeed, (2.45) is a consequence of

$$\left[\hat{\Phi}_0(x), \hat{\Phi}_0(y) \right]_- = i \Delta_m(x - y) \stackrel{\text{def}}{=} \left\langle \Omega_0 \mid \left[\hat{\Phi}_0(x), \hat{\Phi}_0(y) \right]_- \Omega_0 \right\rangle \quad (2.46)$$

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²²As we will learn from the HAAG-RUELLE scattering theory in Section 2.3, it is not that important to give a concrete physical interpretation for $\Phi_0(x)$.

²³We use $x \times y$ as a shorthand notation for $(x - y)^2 < 0$; i.e. for x being space-like to y .

and the fact that

$$\Delta_m(x) = -i(2\pi)^{-3} \int \frac{p^0}{|p^0|} \delta(p^2 - m^2) e^{-ipx} dp, \quad (2.47)$$

being an odd LORENTZ invariant distribution, vanishes for $x \times 0$ (Güttlinger and Rieckers, 1968) (see also Footnote 60).

Exercise 35 Show, for arbitrary $\varphi \in \mathcal{S}(\mathbb{R}^4)$, that

$$\int \hat{\Phi}_0(x) \varphi(x) dx = \sqrt{2\pi} (\hat{a}_0(\check{\varphi}_-) + \hat{a}_0^*(\check{\varphi}_+)),$$

where

$$\check{\varphi}_\pm(p) \stackrel{\text{def}}{=} \left(\frac{\tilde{\varphi}(\pm p)}{2p^0} \right) \Big|_{p^0=\omega_{\mathbf{p}}}$$

and

$$\tilde{\varphi}(p) \stackrel{\text{def}}{=} (2\pi)^{-1} \int \varphi(x) e^{+ipx} dx \quad (\text{FOURIER transform}).$$

By Exercise 35 we easily see that Ω_0 is a **cyclic vector** for the algebra \mathcal{P}_0 generated by $\hat{1} \wedge D_0$ and the smeared field operators $\hat{\Phi}_0(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^4)$, i.e.:

$$\mathcal{P}_0 \Omega \text{ is dense in } \mathcal{H}_0.$$

Obviously, the common domain D_0 has the following invariance properties:²⁴

$$\hat{\Phi}_0(\varphi)D_0 \subset D_0 \supset \hat{U}_0(a, \Lambda)D_0 \quad (2.48)$$

(hence $\mathcal{P}_0\Omega \subset D_0$).

Exercise 36

- (i) Determine the observable for the particle density according to NEWTON and WIGNER.
- (ii) What changes will arise for $\hat{\Phi}_0(x)$, if one defines $\hat{a}_0(\mathbf{p})$ by (2.23)/(2.28) using **anti**-symmetric $\check{f}_n(\mathbf{p}_1, \dots, \mathbf{p}_n)$ (and $\hat{a}_0^*(\mathbf{p})$ by (2.33), again)?
- (iii) Determine the norm²⁵ of $\hat{\Phi}_0(\varphi) \wedge \mathcal{H}_0^{(n)}$ as a function of $n \in \mathbb{Z}_+$.

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²⁴We even have $D_0 = \hat{U}_0(a, \Lambda)D_0$.

²⁵**Hint:** Recall Exercise 35 and study the operators considered in Footnote 19 first.

2.2 WIGHTMAN Theory for Neutral Scalar Fields

2.2.1 WIGHTMAN Axioms

A WIGHTMAN *theory* of a single neutral scalar field $\hat{\Phi}(x)$ is characterized by the following assumptions (WIGHTMAN *axioms*):

0. Assumptions of Relativistic Quantum Theory:

The pure states are given – in the sense of orthodox quantum theory – by the vectors Ψ of some separable,²⁶ complex HILBERT space \mathcal{H} on which a (strongly) continuous, unitary representation $\hat{U}(a, \Lambda)$ of \mathcal{P}_+^\uparrow acts according to the following interpretation:

An ensemble corresponding to $\hat{U}(a, \Lambda)\Psi$ is to be described with respect to the coordinates $x' = \Lambda^{-1}(x - a)$ in exactly the same way as an ensemble corresponding to $\Psi \in \mathcal{H}$ is to be described with respect to the coordinates x .

This representation fulfills the following *spectrum condition*:

For the spectral measure \hat{E} of the observable

$$\hat{P} = \int p \hat{E}(dp)$$

of 4-momentum, uniquely characterized by²⁷

$$\langle \Psi | \hat{U}(a, \mathbb{1}_4)\Psi \rangle = \int e^{ipa} \langle \Psi | \hat{E}(dp)\Psi \rangle \quad \forall \Psi \in \mathcal{H},$$

we have

$$\hat{E}(B) = 0 \text{ for all BOREL } B \subset \mathbb{R}^4 \setminus \overline{V_+},$$

where

$$V_\pm \stackrel{\text{def}}{=} \{x \in \mathbb{R}^4 : xx > 0, \pm x^0 > 0\}.$$

There is a normed vector Ω , unique up to a phase factor, that is invariant under the representation of \mathcal{P}_+^\uparrow :

$$\hat{U}(a, \Lambda)\Omega = \Omega \quad \forall (a, \Lambda) \in \mathcal{P}_+^\uparrow.$$

This vector describes the *vacuum state* of the theory.

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²⁶Actually, separability is a consequence of the separability of the test function space and cyclicity of the vacuum state.

²⁷That such a projection-valued measure \hat{E} on \mathbb{R}^4 exists is guaranteed by the so-called **SNAG theorem** (see (Streater and Wightman, 1989, p. 92) and references given there).

I. Assumptions about the Domain and Continuity of the Field:

The field $\hat{\Phi}(x)$ is a **hermitian operator-valued, tempered generalized function with invariant domain** $D \subset \mathcal{H}$; i.e. a **linear** mapping²⁸

$$\begin{aligned} \hat{\Phi} : \mathcal{S}(\mathbb{R}^4) &\longrightarrow L(D, D) \\ \varphi &\longmapsto \hat{\Phi}(\varphi) = \underbrace{\int \hat{\Phi}(x)\varphi(x) dx}_{\text{formal}} \end{aligned}$$

for which all the

$$\int \langle \Psi | \hat{\Phi}(x)\Psi \rangle \varphi(x) dx \stackrel{\text{def}}{=} \langle \Psi | \hat{\Phi}(\varphi)\Psi \rangle, \Psi \in D,$$

are **continuous** in $\varphi \in \mathcal{S}(\mathbb{R}^4)$, where D has to fulfill the following conditions for $\varphi \in \mathcal{S}(\mathbb{R}^4)$ and $(a, \Lambda) \in \mathcal{P}_+^\uparrow$:

$$\Omega \subset D, \quad \hat{U}(a, \Lambda)D \subset D, \quad \hat{\Phi}(\varphi)D \subset D, \quad \hat{\Phi}(\bar{\varphi}) = \hat{\Phi}(\varphi)^* \wedge D.$$

II. Transformation Law of the Field:

The field transforms according to

$$\hat{U}(a, \Lambda)\hat{\Phi}(x)\hat{U}(a, \Lambda)^{-1} = \hat{\Phi}(\Lambda x + a) \quad \forall (a, \Lambda) \in \mathcal{P}_+^\uparrow.$$

III. Local Commutativity (Microscopic Causality):

The smeared fields $\hat{\Phi}(\varphi_1)$, $\hat{\Phi}(\varphi_2)$ commute whenever the supports of the test functions $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^4)$ are spacelike with respect to each other.²⁹

Formally:

$$x \times y \implies [\hat{\Phi}(x), \hat{\Phi}(y)]_- = 0.$$

Finally, the vacuum vector Ω is required to be **cyclic** with respect to the algebra \mathcal{F}_0 generated by $\hat{1} \wedge D$ and the smeared field operators $\hat{\Phi}(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^4)$. This means:

$$\mathcal{F}_0 \Omega \text{ is dense in } \mathcal{H}.$$

Obviously, all the WIGHTMAN axioms are fulfilled for the free field $\hat{\Phi}_0(x)$.

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²⁸As usual, if X and Y are linear spaces, we denote by $L(X, Y)$ the set of all linear mappings from X into Y . We do no longer assume smearing of the field in the space variables to be sufficient. Actually it can be shown that smearing in x^0 would be sufficient (Borchers, 1964).

²⁹That this condition has to be fulfilled in order to avoid acausal effects even at the microscopic scale, if the field $\hat{\Phi}(x)$ is fully observable, is shown in (Schlieder, 1971).

2.2.2 Remarks on the Choice of the Space of Test Functions

Originally (Wightman, 1956), WIGHTMAN used the SCHWARTZ space

$$\mathcal{D}(\mathbb{R}^4) \stackrel{\text{def}}{=} \{\varphi \in \mathcal{S}(\mathbb{R}^4) : \text{supp}\varphi \text{ compact}\}$$

with the usual topology:

$$\varphi_\nu \rightarrow \varphi \text{ in } \mathcal{D}(\mathbb{R}^4) \text{ if and only if}^{30}$$

$$D_x^\alpha \varphi_\nu(x) \rightarrow D_x^\alpha \varphi(x) \text{ uniformly in } x \in \mathbb{R}^4 \quad \forall \alpha \in \mathbb{Z}_+^4$$

and if there is a compact subset $K \subset \mathbb{R}^4$ outside which all the φ_ν vanish.

Since the FOURIER transform $\tilde{\varphi}$ of a test function $\varphi \in \mathcal{D}(\mathbb{R}^4)$ is always an entire analytic function, the generalized functions on $\mathcal{D}(\mathbb{R}^4)$ are **nonlocalizable**, in general, i.e. there is no notion of support in the ordinary sense for this class of Distributions. In his original program (Wightman, 1956), WIGHTMAN indicated corresponding problems by the remark:

“We shall assume that $F^{(n)}$ has a FOURIER transform.”

Now, the FOURIER transform $\tilde{F}^{(n)}$ of a generalized function $F^{(n)}$ on $\mathcal{D}(\mathbb{R}^4)$ is always well defined on the FOURIER dual $\tilde{\mathcal{D}}(\mathbb{R}^4)$ of $\mathcal{D}(\mathbb{R}^4)$. What WIGHTMAN ment was that his use of the LAPLACE transform of $\tilde{F}^{(n)}$ should be justified. Using the notion of quasi-support for nonlocalizable generalized functions, introduced in (Bümmerstedt and Lücke, 1974) (via *local continuity*, as explained in 3.3.3), and the corresponding definition of LAPLACE transform, introduced in (Lücke, 1984, Sect. 4), there is no problem at all. Another method to justify WIGHTMAN’s results without additional assumptions was presented earlier in (Borchers, 1964, Sect. 5).

The choice of test function space may well be crucial (Wightman, 1981), since:

Nobody could construct a WIGHTMAN field (for the test space $\mathcal{S}(\mathbb{R}^4)$) with nontrivial interaction.

If physical space-time $\mathbb{R}^1 \times \mathbb{R}^3$ is replaced by a toy space-time $\mathbb{R}^1 \times \mathbb{R}^d$ with $d < 3$ then the test space $\mathcal{S}(\mathbb{R}^{d+1})$ is known to be suitable.³¹

E.P. Osipov (hep-th/9608115) claims to be able to construct a field $\hat{\Phi}(x)$ on the physical space-time with nontrivial S-matrix fulfilling all the WIGHTMAN axioms with $\mathcal{S}(\mathbb{R}^4)$ replaced by some **Jaffe space**³² $\mathcal{J}(\mathbb{R}^4)$, where

$$\mathcal{D}(\mathbb{R}^4) \cap \mathcal{J}(\mathbb{R}^4) \text{ is dense in } \mathcal{J}(\mathbb{R}^4) \subset \mathcal{S}(\mathbb{R}^4).$$

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³⁰We use standard multi-index notation: $D_x^\alpha \stackrel{\text{def}}{=} \left(\frac{\partial}{\partial x^0}\right)^{\alpha_0} \cdots \left(\frac{\partial}{\partial x^3}\right)^{\alpha_3}$.

³¹See (Streater and Wightman, 1989, Appendix) for a neat review of the construction of corresponding models. For more details see (Glimm and Jaffe, 1981).

³²See (Jaffe, 1967) for Jaffe’s class of test spaces for **localizable** fields.

Examples of nonlocalizable fields are carefully studied in (Rieckers, 1971). How the condition of microcausality has to be modified for these examples is shown in (Lücke, 1974a).

2.2.3 Mathematical Tools

Theorem 2.2.1 (SCHWARTZ'S Nuclear Theorem) *Let $n_1, n_2 \in \mathbb{N}$ and let*

$$\begin{aligned} F : \mathcal{S}(\mathbb{R}^{n_1}) \times \mathcal{S}(\mathbb{R}^{n_2}) &\longrightarrow \mathbb{C} \\ (\varphi_1, \varphi_2) &\longmapsto F(\varphi_1, \varphi_2) \end{aligned}$$

be linear and continuous in each argument separately. Then there is a one and only one generalized function $F(x_1, x_2)$ on $\mathcal{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ with

$$\int F(x_1, x_2) \varphi_1(x_1) \varphi_2(x_2) dx_1 dx_2 = F(\varphi_1, \varphi_2) \quad \forall (\varphi_1, \varphi_2) \in \mathcal{S}(\mathbb{R}^{n_1}) \times \mathcal{S}(\mathbb{R}^{n_2}).$$

Proof:³³ See (Gelfand and Wilenkin, 1964, Chapter I §1 Nr. 2). ■

Theorem 2.2.1 (together with the Hahn-Banach theorem) implies that³⁴

$$\mathcal{S}(\mathbb{R}^{n_1}) \otimes \mathcal{S}(\mathbb{R}^{n_2}) \text{ is dense in } \mathcal{S}(\mathbb{R}^{n_1+n_2}) \quad \forall n_1, n_2 \in \mathbb{N}. \quad (2.49)$$

Therefore the following Lemma allows iteration (Corollary 2.2.3, below) of the nuclear theorem.

Lemma 2.2.2 *Let $n \in \mathbb{N}$ and let $\{F_\nu\}_{\nu \in \mathbb{Z}_+} \subset \mathcal{S}(\mathbb{R}^n)'$ be such that $\lim_{\nu \rightarrow \infty} F_\nu(\varphi)$ exists for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$F(\varphi) \stackrel{\text{def}}{=} \lim_{\nu \rightarrow \infty} F_\nu(\varphi) \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^n)$$

defines a tempered generalized function on $\mathcal{S}(\mathbb{R}^n)$; i.e. $F \in \mathcal{S}(\mathbb{R}^n)'$.

Proof: See (Gelfand and Schilow, 1962, Ch. I §5 No. 6). ■

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³³See also (Bümmerstedt and Lücke, 1974, Appendix).

³⁴As usual, we denote by $\mathcal{S}(\mathbb{R}^{n_1}) \otimes \mathcal{S}(\mathbb{R}^{n_2})$ the algebraic tensor product of $\mathcal{S}(\mathbb{R}^{n_1})$ and $\mathcal{S}(\mathbb{R}^{n_2})$ realized as the linear span of the set of all $\varphi \in \mathcal{S}(\mathbb{R}^{n_1+n_2})$ of the form

$$\varphi(x_1, x_2) = \varphi_1(x_1) \varphi_2(x_2), \quad \varphi_j \in \mathcal{S}(\mathbb{R}^{n_j}) \text{ for } j = 1, 2.$$

Corollary 2.2.3 *Let $k, n_1, \dots, n_k \in \mathbb{N}$ and let*

$$\begin{aligned} F : \mathcal{S}(\mathbb{R}^{n_1}) \times \dots \times \mathcal{S}(\mathbb{R}^{n_k}) &\longrightarrow \mathbb{C} \\ (\varphi_1, \dots, \varphi_k) &\longmapsto F(\varphi_1, \dots, \varphi_k) \end{aligned}$$

be linear and continuous in each argument separately. Then there is a one and only one generalized function $F \in \mathcal{S}(\mathbb{R}^{n_1+\dots+n_k})'$ with

$$\int F(x_1, \dots, x_k) \varphi_1(x_1) \cdots \varphi_k(x_k) dx_1 \cdots dx_k = F(\varphi_1, \dots, \varphi_k)$$

for all $(\varphi_1, \dots, \varphi_k) \in \mathcal{S}(\mathbb{R}^{n_1}) \times \dots \times \mathcal{S}(\mathbb{R}^{n_k})$.

Given $n \in \mathbb{N}$ and $\varphi \in \mathcal{S}(\mathbb{R}^{4n})$, let us define

$$\left(\hat{K}_\Delta \varphi \right) \left(\frac{x_1 + \dots + x_n}{n}, x_1 - x_2, \dots, x_{n-1} - x_n \right) \stackrel{\text{def}}{=} \varphi(x_1, \dots, x_n)$$

for $x_1, \dots, x_n \in \mathbb{R}^4$. Then \hat{K}_Δ and its inverse \hat{K}_Δ^{-1} are linear continuous mappings of $\mathcal{S}(\mathbb{R}^{4n})$ into itself with

$$\begin{aligned} &\int \left(\hat{K}_\Delta \varphi \right) (X, \xi_1, \dots, \xi_{n-1}) \psi(X, \xi_1, \dots, \xi_{n-1}) dX d\xi_1 \dots d\xi_{n-1} \\ &= \int \varphi(x_1, \dots, x_n) \left(\hat{K}_\Delta^{-1} \psi \right) (x_1, \dots, x_n) dx_1 \cdots dx_n \quad \forall \varphi, \psi \in \mathcal{S}(\mathbb{R}^{4n}). \end{aligned}$$

Therefore, as explained in 2.1.3, $\hat{K}_\Delta F$ is well defined for tempered **generalized** functions F on \mathbb{R}^{4n} .

Lemma 2.2.4 *Let $n \in \mathbb{R}^4$ and let $W \in \mathcal{S}(\mathbb{R}^{4n})'$ fulfill*

$$W(x_1 + a, \dots, x_n + a) = W(x_1, \dots, x_n) \quad \forall a \in \mathbb{R}^4.$$

Then there is a unique generalized function $\mathcal{W} \in \mathcal{S}(\mathbb{R}^{4(n-1)})'$ with

$$W(x_1, \dots, x_n) = \mathcal{W}(x_1 - x_2, \dots, x_{n-1} - x_n),$$

i.e.:

$$\begin{aligned} &\int W(x_1, \dots, x_n) \varphi(x_1, \dots, x_n) dx_1 \cdots dx_n \\ &= \int \mathcal{W}(\xi_1, \dots, \xi_{n-1}) \left(\int \left(\hat{K}_\Delta \varphi \right) (a, \xi_1, \dots, \xi_{n-1}) da \right) d\xi_1 \cdots d\xi_{n-1} \end{aligned}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^{4n})$.

Sketch of proof:³⁵ Let us choose some $\chi \in \mathcal{S}(\mathbb{R}^4)$ with

$$\int \chi(a) da = 1$$

and use the short-hand notation

$$\begin{aligned} \tilde{x} &\stackrel{\text{def}}{=} (x_1, \dots, x_n), & \check{\xi} &\stackrel{\text{def}}{=} (\xi_1, \dots, \xi_{n-1}), \\ d\tilde{x} &\stackrel{\text{def}}{=} dx_1 \cdots dx_n, & d\check{\xi} &\stackrel{\text{def}}{=} d\xi_1 \cdots d\xi_{n-1}. \end{aligned}$$

Then

$$\begin{aligned} &\int W(\tilde{x}) \varphi(\tilde{x}) d\tilde{x} \\ &= \int W(\tilde{x}) \varphi(x_1 - a, \dots, x_n - a) d\tilde{x} \\ &= \int \left(\hat{K}_\Delta W \right) (X, \check{\xi}) \left(\hat{K}_\Delta \varphi \right) (X - a, \check{\xi}) dX d\check{\xi} \\ &= \int \left(\int \left(\hat{K}_\Delta W \right) (X, \check{\xi}) \left(\hat{K}_\Delta \varphi \right) (X - a, \check{\xi}) dX d\check{\xi} \right) \chi(a) da \\ &= \int \left(\hat{K}_\Delta W \right) (X, \check{\xi}) \left(\int \left(\hat{K}_\Delta \varphi \right) (X - a, \check{\xi}) \chi(a) da \right) dX d\check{\xi} \\ &= \int \left(\hat{K}_\Delta W \right) (X, \check{\xi}) \left(\int \left(\hat{K}_\Delta \varphi \right) (a, \check{\xi}) \chi(X - a) da \right) dX d\check{\xi} \\ &= \int \left(\hat{K}_\Delta W \right) (X, \check{\xi}) \left(\int \left(\hat{K}_\Delta \varphi \right) (a, \check{\xi}) \chi(X) da \right) dX d\check{\xi} \\ &= \int \left(\hat{K}_\Delta W \right) (X, \check{\xi}) \chi(X) \left(\int \left(\hat{K}_\Delta \varphi \right) (a, \check{\xi}) da \right) dX d\check{\xi} \end{aligned}$$

and hence

$$\mathcal{W}(\xi_1, \dots, \xi_{n-1}) = \int \left(\hat{K}_\Delta W \right) (X, \xi_1, \dots, \xi_{n-1}) \chi(X) dX. \quad \blacksquare$$

Theorem 2.2.5 (BOCHNER-SCHWARTZ) *Let $\mathcal{W} \in \mathcal{D}(\mathbb{R}^4)$ be **positive semi definite**, i.e. fulfill*

$$\int \mathcal{W}(x - y) \overline{\varphi(x)} \varphi(y) dx dy \geq 0 \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^4).$$

Then there are a (unique) positive BOREL measure μ on \mathbb{R}^4 and some $k \in \mathbb{Z}_+$ with

$$\int (1 + \|p\|)^{-k} \mu(dp) < \infty$$

*and*³⁶

$$\int \widetilde{\mathcal{W}}(q) \psi(q) dq = \int \psi(q) \mu(dq) \quad \forall \psi \in \widetilde{\mathcal{D}}(\mathbb{R}^4).$$

Hence \mathcal{W} is the restriction to $\mathcal{D}(\mathbb{R}^4)$ of a tempered generalized function.

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³⁵The interchanges of (ordinary and/or formal) integration may be easily justified by intermediate regularization of the involved generalized functions. An alternative, more indirect, proof can be found in (Streater and Wightman, 1989, p. 39/40).

³⁶Of course, $\widetilde{\mathcal{D}}(\mathbb{R}^4)$ means the FOURIER dual of $\mathcal{D}(\mathbb{R}^4)$: $\widetilde{\mathcal{D}}(\mathbb{R}^4) \stackrel{\text{def}}{=} \{\tilde{\varphi} : \varphi \in \mathcal{D}(\mathbb{R}^4)\}$.

Proof: See (Gelfand and Wilenkin, 1964, Ch. II §3). ■

Lemma 2.2.6 *Let $n \in \mathbb{R}^4$ and let $W \in \mathcal{S}(\mathbb{R}^{4n})'$ fulfill*

$$W(x_1 + a, \dots, x_n + a) = W(x_1, \dots, x_n) \quad \forall a \in \mathbb{R}^4$$

as well as

$$\text{supp} \tilde{\chi} \subset \mathbb{R}^4 \setminus \overline{V_+} \implies \int W(x_1, \dots, x_\nu + a, \dots, x_n + a) \chi(a) da = 0$$

for all $\chi \in \mathcal{S}(\mathbb{R}^4)$ and $\nu = 2, \dots, n$. Then for the FOURIER transform

$$\widetilde{\mathcal{W}}(q_1, \dots, q_{n-1}) \stackrel{\text{def}}{=} (2\pi)^{-2(n-1)} \int \mathcal{W}(\xi_1, \dots, \xi_{n-1}) e^{i(\xi_1 q_1 + \dots + \xi_{n-1} q_{n-1})} d\xi_1 \cdots d\xi_{n-1}$$

of the generalized function

$$\mathcal{W}(\xi_1, \dots, \xi_{n-1}) = W(\xi_1 + \dots + \xi_{n-1}, \xi_2 + \dots + \xi_{n-1}, \dots, \xi_{n-1}, 0)$$

given by Lemma 2.2.4 we have:

$$\text{supp} \widetilde{\mathcal{W}} \subset \overline{V_+} \times \cdots \times \overline{V_+}.$$

Sketch of proof: By Lemma 2.2.4 we have for all $\nu \in \{2, \dots, n\}$

$$W(x_1, \dots, x_{\nu+1} + a, \dots, x_n + a) = \mathcal{W}(\xi_1, \dots, \xi_\nu - a, \dots, \xi_{\nu-1}) \quad \forall a \in \mathbb{R}^4$$

and hence for all $\chi \in \mathcal{S}(\mathbb{R}^4)$ with $\text{supp} \tilde{\chi} \subset \mathbb{R}^4 \setminus \overline{V_+}$

$$\begin{aligned} 0 &= \int \left(\int \mathcal{W}(\xi_1, \dots, \xi_\nu - a, \dots, \xi_{\nu-1}) e^{i(\xi_1 q_1 + \dots + \xi_{\nu-1} q_{\nu-1})} d\xi_1 \cdots d\xi_{\nu-1} \right) \chi(a) da \\ &= \left(\int \mathcal{W}(\xi_1, \dots, \xi_{\nu-1}) e^{i(\xi_1 q_1 + \dots + \xi_{\nu-1} q_{\nu-1})} d\xi_1 \cdots d\xi_{\nu-1} \right) \int \chi(a) e^{i q_\nu a} da \\ &= (2\pi)^{2(n-1)} \widetilde{\mathcal{W}}(q_1, \dots, q_{n-1}) \tilde{\chi}(q_\nu). \quad \blacksquare \end{aligned}$$

Lemma 2.2.7 *Let $1 < n \in \mathbb{N}$ and let $\mathcal{W}(\xi_1, \dots, \xi_{n-1}) \in \mathcal{S}(\mathbb{R}^{4(n-1)})$ be such that*

$$\text{supp} \widetilde{\mathcal{W}} \subset \overline{V_+} \times \cdots \times \overline{V_+}.$$

Then there is one and only one holomorphic function $(\mathcal{L}\widetilde{\mathcal{W}})(z_1, \dots, z_{n-1})$ on

$$\mathcal{T}_{n-1} \stackrel{\text{def}}{=} \{(z_1, \dots, z_{n-1}) \in \mathbb{C} : \Im(z_\nu) \in V_- \text{ for } \nu = 1, \dots, n-1\},$$

fulfilling the condition³⁷

$$\begin{aligned} &\int (\mathcal{L}\widetilde{\mathcal{W}})(\xi_1 + i\eta_1, \dots, \xi_{n-1} + i\eta_{n-1}) \tilde{\varphi}(\xi_1, \dots, \xi_{n-1}) d\xi_1 \cdots d\xi_{n-1} \\ &= \int \widetilde{\mathcal{W}}(q_1, \dots, q_{n-1}) (e^{q_1 \eta_1 + \dots + q_{n-1} \eta_{n-1}} \varphi(q_1, \dots, q_{n-1})) dq_1 \cdots dq_{n-1}. \end{aligned}$$

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³⁷Note that

$$\int F(q_1, \dots, q_{n-1}) \tilde{\varphi}(q_1, \dots, q_{n-1}) dq_1 \cdots dq_{n-1} = \int \tilde{F}(q_1, \dots, q_{n-1}) \varphi(q_1, \dots, q_{n-1}) dq_1 \cdots dq_{n-1}$$

for all $F \in \mathcal{S}(\mathbb{R}^{4(n-1)})'$ and $\varphi \in \mathcal{S}(\mathbb{R}^{4(n-1)})$.

This so-called LAPLACE *transform*

$$\begin{aligned} & \left(\mathcal{L}\widetilde{\mathcal{W}} \right) (\xi_1 + i\eta_1, \dots, \xi_{n-1} + i\eta_{n-1}) \\ &= \underbrace{\int \widetilde{\mathcal{W}}(q_1, \dots, q_{n-1}) e^{-i(q_1(\xi_1 + i\eta_1) + \dots + q_{n-1}(\xi_{n-1} + i\eta_{n-1}))} dq_1 \cdots dq_{n-1}}_{\text{formal}} \end{aligned}$$

of $\widetilde{\mathcal{W}}$ fulfills³⁸

$$\lim_{V_- \ni \eta_1, \dots, \eta_{n-1} \rightarrow 0} \int \left(\mathcal{L}\widetilde{\mathcal{W}} \right) (\xi_1 + i\eta_1, \dots, \xi_{n-1} + i\eta_{n-1}) \psi(\xi_1, \dots, \xi_{n-1}) d\xi_1 \cdots d\xi_{n-1} \\ = \int \mathcal{W}(\xi_1, \dots, \xi_{n-1}) \psi(\xi_1, \dots, \xi_{n-1}) d\xi_1 \cdots d\xi_{n-1} \quad \forall \psi \in \mathcal{S}(\mathbb{R}^{4(n-1)}).$$

Sketch of proof: Choose some $\chi \in C^\infty(\mathbb{R}^{4(n-1)})$ with

$$\chi(\check{\xi}) = \begin{cases} 1 & \text{if } \|\check{\xi} - \check{\xi}'\| < 1 \text{ for all } \check{\xi}' \in V_+ \times \cdots \times V_+ \\ 0 & \text{if } \|\check{\xi} - \check{\xi}'\| > 2 \text{ for some } \check{\xi}' \in V_+ \times \cdots \times V_+ \end{cases}$$

and define

$$\begin{aligned} & \left(\mathcal{L}\widetilde{\mathcal{W}} \right) (\xi_1 + i\eta_1, \dots, \xi_{n-1} + i\eta_{n-1}) \\ & \stackrel{\text{def}}{=} \int \widetilde{\mathcal{W}}(q_1, \dots, q_{n-1}) \underbrace{\left(e^{-i(q_1(\xi_1 + i\eta_1) + \dots + q_{n-1}(\xi_{n-1} + i\eta_{n-1}))} \chi(q_1, \dots, q_{n-1}) \right)}_{\in \mathcal{S}(\mathbb{R}^{4(n-1)})} dq_1 \cdots dq_{n-1} \end{aligned}$$

for $\eta_1, \dots, \eta_{n-1} \in V_-$. ■

The following theorem shows that $\left(\mathcal{L}\widetilde{\mathcal{W}} \right) (\xi_1 + i\eta_1, \dots, \xi_{n-1} + i\eta_{n-1})$ is already fixed by its (distributional) boundary values on any open subset \mathcal{O} of \mathbb{R}^{n-1} .

Theorem 2.2.8 (Edge-of-the-Wedge) *Let $n \in \mathbb{N}$, let $\check{\mathcal{O}}$ be an open subset of \mathbb{C}^n for which $\mathcal{O} \stackrel{\text{def}}{=} \check{\mathcal{O}} \cap \mathbb{R}^n \neq \emptyset$, let \mathcal{C} be an open convex cone in \mathbb{R}^n with apex at 0, and let L be a holomorphic function on*

$$\mathcal{B} \stackrel{\text{def}}{=} (\mathbb{R}^n + i\mathcal{C}) \cap \check{\mathcal{O}}$$

such that

$$\lim_{\mathcal{C} \ni y \rightarrow 0} \int L(x + iy) \varphi(x) dx = 0 \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^n) \text{ with } \text{supp} \varphi \subset \mathcal{O}.$$

Then

$$L(x + iy) = 0 \quad \forall x + iy \in \mathcal{B}.$$

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³⁸Compare (Lücke, 1984, Theorem 4).

Proof: See (Streater and Wightman, 1989, Theorem 2-17). ■

Corollary 2.2.9 *Let $1 < n \in \mathbb{N}$ and $\mathcal{W} \in \mathcal{S}(\mathbb{R}^{4(n-1)})'$. If*

$$\text{supp}\widetilde{\mathcal{W}} \subset \overline{V}_+ \times \dots \times \overline{V}_+$$

then either $\mathcal{W} = 0$ or $\text{supp}\mathcal{W} = \mathbb{R}^{4(n-1)}$.

Proof: Direct consequence of Lemma 2.2.7 and Theorem 2.2.8. ■

2.2.4 Some Standard Results

By Corollary 2.2.3, for every $n \in \mathbb{N}$ and every $\Psi \in D$ there is a unique generalized function

$$\langle \Psi | \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n) \Psi \rangle \in \mathcal{S}(\mathbb{R}^{4n})'$$

with

$$\int \langle \Psi | \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n) \Psi \rangle \varphi_1(x_1) \cdots \varphi_n(x_n) dx_1 \cdots dx_n = \langle \Psi | \hat{\Phi}(\varphi_1) \cdots \hat{\Phi}(\varphi_n) \Psi \rangle$$

for all $\varphi_1, \dots, \varphi_n \in \mathcal{S}(\mathbb{R}^4)$. Thus, especially, the so-called *n -point functions*

$$W(x_1, \dots, x_n) = \langle \Omega | \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n) \Omega \rangle, \quad (2.50)$$

are well defined as generalized functions on $\mathcal{S}(\mathbb{R}^{4n})$. The relativistic transformation law for $\hat{\Phi}(x)$ and the invariance of the vacuum imply

$$\begin{aligned} & \int W(\Lambda x_1 + a, \dots, \Lambda x_n + a) \varphi_1(x_1) \cdots \varphi_n(x_n) dx_1 \cdots dx_n \\ &= \langle \Omega | \hat{U}(a, \Lambda) \hat{\Phi}(\varphi_1) \underbrace{\hat{U}(a, \Lambda)^{-1} \hat{U}(a, \Lambda)}_{=\hat{1}} \cdots \hat{U}(a, \Lambda) \hat{\Phi}(\varphi_n) \underbrace{\hat{U}(a, \Lambda)^{-1} \Omega}_{=\Omega} \rangle \end{aligned}$$

and, by the same reasoning,

$$\begin{aligned} & \int W(x_1, \dots, x_\nu + a, \dots, x_n + a) \varphi_1(x_1) \cdots \varphi_n(x_n) dx_1 \cdots dx_n \\ &= \langle \Omega | \hat{\Phi}(\varphi_1) \cdots \hat{U}(a) \hat{\Phi}(\varphi_2) \cdots \hat{\Phi}(\varphi_n) \Omega \rangle \end{aligned}$$

for all $(a, \Lambda) \in \mathcal{P}_+^\uparrow$, $\varphi_1, \dots, \varphi_n \in \mathcal{S}(\mathbb{R}^4)$, and $\nu = 2, \dots, n$. By (2.49), this implies

$$W(x_1, \dots, x_n) \stackrel{\text{def}}{=} W(\Lambda x_1 + a, \dots, \Lambda x_n + a) \quad \forall n \in \mathbb{N}, (a, \Lambda) \in \mathcal{P}_+^\uparrow \quad (2.51)$$

and, thanks to the spectrum condition,³⁹

$$\text{supp}\tilde{\chi} \subset \mathbb{R}^4 \setminus \overline{V}_+ \implies \int W(x_1, \dots, x_\nu + a, \dots, x_n + a) \chi(a) da = 0.$$

Therefore, according to Lemma 2.2.4 and Lemma 2.2.6, for every natural number $n > 1$ there is a generalized function $\mathcal{W} \in \mathcal{S}(\mathbb{R}^{4(n-1)})$ with

$$W(x_1, \dots, x_n) = \mathcal{W}(x_1 - x_2, \dots, x_{n-1} - x_n) \quad (2.52)$$

and

$$\text{supp}\tilde{\mathcal{W}} \subset \overline{V}_+ \times \dots \times \overline{V}_+. \quad (2.53)$$

Exercise 37 Show the following:

- (i) For every $\varphi \in \mathcal{S}(\mathbb{R}^4)$ there is a unique⁴⁰ operator

$$\int \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n) \varphi(x_1, \dots, x_n) dx_1 \cdots dx_n \in L(D, \mathcal{H})$$

with

$$\begin{aligned} & \left\langle \Psi \left| \left(\int \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n) \varphi(x_1, \dots, x_n) dx_1 \cdots dx_n \right) \Psi \right\rangle \right. \\ &= \int \left\langle \Psi \left| \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n) \Psi \right\rangle \varphi(x_1, \dots, x_n) dx_1 \cdots dx_n \quad \forall \Psi \in D. \end{aligned}$$

- (ii) For every $\Psi \in D$

$$\int \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n) \varphi(x_1, \dots, x_n) dx_1 \cdots dx_n \Psi$$

depends strongly continuously on $\varphi \in \mathcal{S}(\mathbb{R}^{4n})$.

- (iii) For all $\varphi_1, \dots, \varphi_n \in \mathcal{S}(\mathbb{R}^4)$

$$\int \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n) \varphi_1 \cdots \varphi_n dx_1 \cdots dx_n = \hat{\Phi}(\varphi_1) \cdots \hat{\Phi}(\varphi_n).$$

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³⁹Note that $\int \hat{U}(a) \chi(a) da = \int \tilde{\chi}(p) \hat{E}(dp)$.

⁴⁰Recall that $\langle \Phi | \hat{A} \Psi \rangle = \frac{1}{4} \sum_{k=0}^3 \langle \Phi + i^k \Psi | \hat{A} (\Phi + i^k \Psi) \rangle \quad \forall \Phi, \Psi \in D, \hat{A} \in L(D, \mathcal{H})$.

By Exercise 37, without any restriction of generality, we may assume⁴¹

$$\int \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n) \varphi(x_1, \dots, x_n) dx_1 \cdots dx_n D \subset D$$

and

$$\begin{aligned} & \left(\int \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n) \varphi(x_1, \dots, x_n) dx_1 \cdots dx_n \right)^* \wedge D \\ & = \int \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n) \overline{\varphi(x_n, \dots, x_1)} dx_1 \cdots dx_n \end{aligned} \quad (2.54)$$

for all $n \in \mathbb{N}$ and all $\varphi \in \mathcal{S}(\mathbb{R}^{4n})$.

Obviously, by the GNS technique (recall Footnote 57 of Chapter 1), a WIGHTMAN theory with $D = \mathcal{F}_0\Omega$ can be reconstructed from its n -point functions, up to unitary equivalence.⁴²

Lemma 2.2.10 *In a field theory of the type described in 2.2.1, with $\dim \mathcal{H} > 1$, the field operator $\hat{\Phi}(x)$ cannot be defined pointwise for $x \in \mathbb{R}^4$.*

Proof.⁴³ Obviously, the generalized function $\mathcal{W}(\xi) \in \mathcal{S}(\mathbb{R}^4)'$ associated with the 2-point function is positive semi-definite. Hence, by Theorem 2.2.5, there is a polynomially bounded positive BOREL measure μ on \mathbb{R}^4 with

$$W(x_1, x_2) = \mathcal{W}(x_1 - x_2) = \int e^{ip(x_1 - x_2)} \mu(dp).$$

By (2.51), μ has to be LORENTZ covariant.⁴⁴

$$\mu(B) = \mu(\Lambda B) \quad \forall \text{BOREL } B \in \mathbb{R}^4, \Lambda \in L_+^\uparrow.$$

Let us assume that $\hat{\Phi}(x)$ is well defined for every $x \in \mathbb{R}^4$. Then $\mu(\mathbb{R}^4) = \mathcal{W}(0)$ must be finite. Hence there is a number $\mu_0 \geq 0$ with

$$\mu(B) = \begin{cases} \mu_0 & \text{if } 0 \in B \\ 0 & \text{else} \end{cases} \quad \forall \text{BOREL } B \in \mathbb{R}^4.$$

This implies

$$\left\langle \hat{\Phi}(x_1)\Omega \mid \hat{\Phi}(x_2)\Omega \right\rangle = W(x_1, x_2) = \mathcal{W}(x_1 - x_2) = \mu_0 \quad \forall x_1, x_2 \in \mathbb{R}^4$$

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⁴¹Here we use

$$\begin{aligned} & \left(\int \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_{n_1}) \varphi_1(x_1, \dots, x_{n_1}) dx_1 \cdots dx_{n_1} \right) \left(\int \hat{\Phi}(y_1) \cdots \hat{\Phi}(y_{n_2}) \varphi_2(y_1, \dots, y_{n_2}) dy_1 \cdots dy_{n_2} \right) \\ & = \int \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_{n_1}) \hat{\Phi}(y_1) \cdots \hat{\Phi}(y_{n_2}) \varphi_1(x_1, \dots, x_{n_1}) \varphi_2(y_1, \dots, y_{n_2}) dx_1 \cdots dx_{n_1} dy_1 \cdots dy_{n_2} \end{aligned}$$

of course.

⁴²See (Streater and Wightman, 1989, Sect. 3-4) for full details.

⁴³The main part of this proof, which uses neither microcausality nor the spectrum condition, was presented in (Jaffe, 1967, Introduction).

⁴⁴As usual we denote by L_+^\uparrow the set of all LORENTZ transformations Λ with $(0, \Lambda) \in \mathcal{P}_+^\uparrow$.

even though

$$\left\| \hat{\Phi}(x)\Omega \right\| = \left\| \hat{U}(x)\hat{\Phi}(0)\Omega \right\| = \left\| \hat{\Phi}(0)\Omega \right\| \quad \forall x \in \mathbb{R}^4.$$

This, in turn, implies

$$\hat{\Phi}(x)\Omega = \hat{\Phi}(0)\Omega \quad \forall x \in \mathbb{R}^4$$

and hence

$$\hat{U}(a, \Lambda)\hat{\Phi}(x)\Omega = \hat{\Phi}(x)\Omega \quad \forall (a, \Lambda) \in \mathcal{P}_+^\uparrow, x \in \mathbb{R}^4.$$

Thanks to uniqueness of the vacuum state, therefore,

$$\hat{\Phi}(x)\Omega \sim \Omega \quad \forall x \in \mathbb{R}^4.$$

By cyclicity of the Ω , finally, this gives $\dim \mathcal{H} = 1$. Since this contradicts the assumptions of the lemma, $\hat{\Phi}(x)$ cannot be well defined for every $x \in \mathbb{R}^4$. ■

For the theory of a single neutral scalar field the theorem on the connection between **spin and statistics** becomes especially simple:⁴⁵

Theorem 2.2.11 *There is no quantum field $\hat{\Phi}(x) \neq 0$, not necessarily fulfilling microcausality but all the other assumptions of 2.2.1, for which*

$$x \times y \implies \hat{\Phi}(x)\hat{\Phi}(y) + \hat{\Phi}(y)\hat{\Phi}(x) = 0.$$

Sketch of proof: Thanks to (2.51) and Lemma 2.2.4 there are L_+^\uparrow -invariant generalized functions $F(\xi), G(\xi) \in \mathcal{S}(\mathbb{R}^4)'$ with

$$F(x - y) = \left\langle \Omega \mid \hat{\Phi}(x)\hat{\Phi}(y) \right\rangle - \left\langle \Omega \mid \hat{\Phi}(y)\hat{\Phi}(x) \right\rangle$$

and

$$G(x - y) = \left\langle \Omega \mid \hat{\Phi}(x)\hat{\Phi}(y) \right\rangle + \left\langle \Omega \mid \hat{\Phi}(y)\hat{\Phi}(x) \right\rangle.$$

Since F is also odd, we have⁴⁶

$$\xi \times 0 \implies F(\xi) = 0.$$

Now, assume local anticommutativity:

$$x \times y \implies \hat{\Phi}(x)\hat{\Phi}(y) + \hat{\Phi}(y)\hat{\Phi}(x) = 0.$$

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⁴⁵Actually, **strict** local anti-commutativity is not necessary (Lücke, 1979). See also (Guido and Longo, 1995) and (Davidson, 1997) for a purely algebraic version using strict locality.

⁴⁶Recall the reasoning for (2.45). For general fields one has to exploit the BHW theorem (Theorem 2.2.19, below).

Then, because of

$$\mathcal{W}(x - y) = \left\langle \Omega \mid \hat{\Phi}(x)\hat{\Phi}(y) \right\rangle = \frac{1}{2} (F(x - y) + G(x - y)) ,$$

we have

$$\xi \times 0 \implies \mathcal{W}(\xi) = 0 .$$

By (2.53) and Corollary 2.2.9, therefore, $\mathcal{W} = 0$ and hence $\hat{\Phi}(\varphi)\Omega = 0$ for all $\varphi \in \mathcal{S}(\mathbb{R}^4)$. By cyclicity of Ω , this implies $\hat{\Phi}(x) = 0$. ■

Theorem 2.2.12 (Reeh-Schlieder) *Let $\hat{\Phi}(x)$ fulfill the assumptions of 2.2.1 with possible exception of microcausality⁴⁷ and let $\mathcal{O} \neq \emptyset$ be an open subset of \mathbb{R}^4 . Then the vacuum state Ω is cyclic with respect to the algebra $\mathcal{F}_0(\mathcal{O}) \subset L(D, D)$ generated by $\hat{1} \wedge D$ and all $\hat{\Phi}(\varphi)$ with $\varphi \in \mathcal{S}(\mathbb{R}^4)$.*

Sketch of proof: By cyclicity of Ω with respect to $\mathcal{F}_0 = \mathcal{F}_0(\mathbb{R}^4)$ it is sufficient to prove⁴⁸

$$\begin{aligned} \left\langle \Psi \mid \hat{\Phi}(\varphi_1) \cdots \hat{\Phi}(\varphi_n) \Omega \right\rangle &= 0 \quad \forall \varphi_1, \dots, \varphi_n \in \mathcal{S}(\mathcal{O}) \\ \implies \left\langle \Psi \mid \hat{\Phi}(\varphi_1) \cdots \hat{\Phi}(\varphi_n) \Omega \right\rangle &= 0 \quad \forall \varphi_1, \dots, \varphi_n \in \mathcal{S}(\mathbb{R}^4) \end{aligned} \quad (2.55)$$

for all $\Psi \in D$ and $n \in \mathbb{N}$:

By appropriate transformation of coordinates we get a generalized function $L(\xi_1, \dots, \xi_n) \in \mathcal{S}(\mathbb{R}^{4n})'$ with

$$L(-x_1, x_1 - x_2, \dots, x_{n-1} - x_n) = \left\langle \Psi \mid \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n) \Omega \right\rangle$$

and

$$\text{supp} L \neq \mathbb{R}^{4n} ,$$

thanks to the assumption in (2.55). On the other hand, we have

$$\text{supp} \tilde{L} \subset \overline{V}_+ \times \dots \times \overline{V}_+ ,$$

by essentially the same reasoning as for (2.53). Therefore⁴⁹ Corollary 2.2.9 implies $L = 0$, hence the r.h.s. of (2.55). ■

Theorem 2.2.12 “can be interpreted as meaning that it is difficult to isolate a system described by fields from outside effects.”

(Streater and Wightman, 1989, p. 139)

⁴⁷Strict localizability is essential, however.

⁴⁸As usual, $\mathcal{S}(\mathcal{O})$ denotes the (topological) subspace of all $\varphi \in \mathcal{S}(\mathbb{R}^4)$ with $\text{supp} \varphi \subset \mathcal{O}$.

⁴⁹Note that translation of the FOURIER transform corresponds to multiplication of the original function by some function with constant modulus one.

Corollary 2.2.13 *Let all the assumptions of Theorem 2.2.12 be fulfilled. Moreover, let $\hat{A} \in L(D, \mathcal{H})$ commute with $\mathcal{F}(\mathcal{O})$ in the sense that⁵⁰*

$$\langle \Psi | \hat{A}\hat{C}\Psi \rangle = \langle \hat{C}^*\Psi | \hat{A}\Psi \rangle \quad \forall \Psi \in D, \hat{C} \in \mathcal{F}(\mathcal{O})$$

and assume $D \subset D_{\hat{A}^*}$. Then

$$\hat{A}\Omega = 0 \implies \hat{A} = 0.$$

Proof: If $\hat{A}\Omega = 0$ then we have

$$\langle \Psi | \hat{A}\hat{C}\Omega \rangle = \langle \hat{C}^*\Psi | \hat{A}\Omega \rangle = 0,$$

for all $\Psi \in D$ and $\hat{C} \in \mathcal{F}(\mathcal{O})$. By the Reeh-Schlieder theorem and thanks to $D \in D_{\hat{A}^*}$ this implies $\hat{A} = 0$. ■

Corollary 2.2.14 *Let all the assumptions of Corollary 2.2.13 be fulfilled and let \hat{A} be positive in the sense that*

$$\langle \Psi | \hat{A}\Psi \rangle \geq 0 \quad \forall \Psi \in D.$$

Then

$$\langle \Omega | \hat{A}\Omega \rangle = 0 \implies \hat{A} = 0.$$

Sketch of proof: Assume $\langle \Omega | \hat{A}\Omega \rangle = 0$. Since \hat{A} is positive, it has a positive self-adjoint extension $\hat{\hat{A}}$ (FRIEDRICHS' *theorem*, see e.g. (Yosida, 1971, Ch. XI §7)). With this extension we have

$$\left\langle \sqrt{\hat{\hat{A}}}\Omega \mid \sqrt{\hat{\hat{A}}}\Omega \right\rangle = \langle \Omega \mid \hat{\hat{A}}\Omega \rangle = 0$$

and hence

$$\hat{A}\Omega = \sqrt{\hat{A}} \left(\sqrt{\hat{\hat{A}}}\Omega \right) = 0.$$

By Corollary 2.2.13, this implies $\hat{A} = 0$. ■

Remark: Corollary 2.2.14 shows that there cannot be any local positive energy density or positive 0-component of a local current density with vanishing vacuum expectation value in a field theory of the type described in 2.2.1.

⁵⁰This condition is obviously fulfilled for $\hat{A} \in \mathcal{F}_0(\mathcal{O}')$ with $\mathcal{O}' \times \mathcal{O}$.

Theorem 2.2.15 (Borchers) *Let $\hat{\Phi}(x)$ fulfill the assumptions of 2.2.1 and let $\Psi \in \mathcal{H}$. Then⁵¹*

$$\left(\hat{U}(a)\Psi = \Psi \quad \forall a \in \mathbb{R}^4 \right) \implies \Psi \sim \Omega.$$

Proof: See (Borchers, 1962, Theorem 3). ■

Corollary 2.2.16 *Let $\hat{\Phi}(x)$ fulfill the assumptions of 2.2.1. Then \mathcal{F}_0 is **irreducible** in the sense that every $\hat{C} \in \mathcal{L}(\mathcal{H})$ with*

$$\left\langle \Psi \mid \hat{C}\hat{\Phi}(\varphi)\Psi \right\rangle = \left\langle \hat{\Phi}(\varphi)^*\Psi \mid \hat{C}\Psi \right\rangle \quad \forall \Psi \in D, \varphi \in \mathcal{S}(\mathbb{R}^4) \quad (2.56)$$

must be a multiple of $\hat{1}$.

Sketch of proof: Let $\hat{C} \in \mathcal{L}(\mathcal{H})$ fulfill (2.56). Then it is sufficient to prove

$$\hat{C}\Omega = c\Omega \quad (2.57)$$

for some $c \in \mathbb{C}$ since the latter implies

$$\begin{aligned} \left\langle \Psi \mid \hat{C}\hat{A}\Omega \right\rangle &\stackrel{(2.56)}{=} \left\langle \hat{A}^*\Psi \mid \hat{C}\Omega \right\rangle \\ &\stackrel{(2.57)}{=} c \left\langle \Psi \mid \hat{A}\Omega \right\rangle \quad \forall \Psi \in D, \hat{A} \in \mathcal{F}_0 \end{aligned}$$

and hence $\hat{C} = c\hat{1}$, thanks to cyclicity of Ω . In order to prove (2.57) it is sufficient, by Theorem 2.2.15, to show

$$\hat{U}(a)\hat{C}\Omega = \hat{C}\Omega \quad \forall a \in \mathbb{R}^4$$

which, by cyclicity and translation invariance⁵² of Ω , is equivalent to

$$\left\langle \hat{\Phi}(\varphi_1) \cdots \hat{\Phi}(\varphi_n)\Omega \mid \hat{U}(a)\hat{C}\Omega \right\rangle$$

being independent of $a \in \mathbb{R}^4$ for all $n \in \mathbb{N}$ and $\varphi_1 \dots \varphi_n \in \mathcal{S}(\mathbb{R}^4)$. The latter,

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⁵¹Let \mathcal{T} denote the subspace of all translational invariant vectors. Obviously, then, \mathcal{T} is invariant under \mathcal{P}_+^\uparrow . Therefore, if we already knew \mathcal{T} to be finite dimensional the statement of Theorem 2.2.15 were a simple consequence of the fact that there are no non-trivial unitary representations of L_+^\uparrow .

⁵²Translation invariance of Ω ensures that also $\left\langle \hat{1}\Omega \mid \hat{U}(a)\Omega \right\rangle$ is a -independent.

however, is an easy consequence of (2.56) and the WIGHTMAN axioms:

$$\begin{aligned}
& \int_{\overline{V_+}} e^{+iap} \left\langle \hat{\Phi}(\varphi_1) \cdots \hat{\Phi}(\varphi_n) \Omega \mid \hat{E}(dp) \hat{C} \Omega \right\rangle \\
&= \left\langle \hat{\Phi}(\varphi_1) \cdots \hat{\Phi}(\varphi_n) \Omega \mid \hat{U}(a) \hat{C} \Omega \right\rangle \\
&= \left\langle \left(\hat{U}(-a) \hat{\Phi}(\varphi_1) \hat{U}(a) \right) \cdots \left(\hat{U}(-a) \hat{\Phi}(\varphi_n) \hat{U}(a) \right) \Omega \mid \hat{C} \Omega \right\rangle \\
&= \left\langle \Omega \mid \hat{C} \left(\hat{U}(-a) \hat{\Phi}(\varphi_n) \hat{U}(a) \right)^* \cdots \left(\hat{U}(-a) \hat{\Phi}(\varphi_1) \hat{U}(a) \right)^* \Omega \right\rangle \\
&= \left\langle \hat{C}^* \Omega \mid \hat{U}(-a) \left(\hat{\Phi}(\varphi_1) \cdots \hat{\Phi}(\varphi_n) \right)^* \Omega \right\rangle \\
&= \int_{\overline{V_+}} e^{-iap} \left\langle \hat{C}^* \Omega \mid \hat{E}(dp) \left(\hat{\Phi}(\varphi_1) \cdots \hat{\Phi}(\varphi_n) \right)^* \Omega \right\rangle. \quad \blacksquare
\end{aligned}$$

In the following we shall denote by $\mathcal{F}(\mathcal{O})$, \mathcal{O} any open subset of \mathbb{R}^4 , the algebra generated by $\hat{1} \wedge D$ and all operators \hat{A} of the form

$$\hat{A} = \int \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n) \varphi(x_1, \dots, x_n) dx_1 \cdots dx_n$$

(recall Exercise 37) with $n \in \mathbb{N}$ and $\varphi \in \mathcal{S}(\mathcal{O} \times \cdots \times \mathcal{O})$.

Exercise 38 Show for the free field $\hat{\Phi}(x) = \hat{\Phi}_0(x)$, described in 2.1, that

$$\left. \begin{array}{l} \hat{A} \in \mathcal{F}(\mathcal{O}) \\ [\hat{A}, \mathcal{F}(\mathcal{O})]_- = 0 \end{array} \right\} \implies \hat{A} \sim \hat{1} \wedge D$$

holds for every bounded open set $\mathcal{O} \subset \mathbb{R}^4$.

Lemma 2.2.17 *Let all the assumption of 2.2.1 be fulfilled.⁵³ Then*

$$\hat{\Phi}(\varphi_1) \cdots \hat{\Phi}(\varphi_n) \Omega \in \hat{E}(\overline{V_+} \cap (\text{supp} \tilde{\varphi}_1 + \dots + \text{supp} \tilde{\varphi}_n)) \mathcal{H}$$

holds for all $n \in \mathbb{N}$ and $\varphi_1, \dots, \varphi_n \in \mathcal{S}(\mathbb{R}^4)$.

Sketch of proof: By

$$\hat{U}(a) = \int e^{ipa} \hat{E}(dp) \tag{2.58}$$

and the basic relation

$$\delta(p - p') = (2\pi)^{-4} \int e^{i(p-p')a} da \tag{2.59}$$

⁵³Actually, microcausality will not be used in the proof of Lemma 2.2.17.

of FOURIER calculus we have

$$\hat{E}(B) = (2\pi)^{-4} \int_B \left(\int \hat{U}(a) e^{-ipa} da \right) dp \quad \forall \text{BOREL } B \subset \mathbb{R}^4. \quad (2.60)$$

By the transformation behaviour of $\hat{\Phi}(x)$ and translation invariance of Ω this implies

$$\begin{aligned} & \hat{E}(B) \hat{\Phi}(\varphi_1) \cdots \hat{\Phi}(\varphi_n) \Omega \\ &= (2\pi)^{-4} \int \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n) \left(\int_B \left(\int \varphi_1(x_1 - a) \cdots \varphi_n(x_n - a) e^{-ipa} da \right) d \right) dx_1 \cdots dx_n \Omega \\ & \quad \forall \text{BOREL } B \subset \mathbb{R}^4. \end{aligned}$$

Since $\int \varphi_1(x_1 - a) \cdots \varphi_n(x_n - a) e^{-ipa} da$ can only be nonzero for $p \in \text{supp}\tilde{\varphi}_1 + \dots + \text{supp}\tilde{\varphi}_n$, this together with the spectrum condition implies the statement of the lemma. ■

Theorem 2.2.18 (Jost-Schroer) *Let \mathcal{H} , $\hat{U}(a, \Lambda)$, Ω , D and $\hat{\Phi}(x)$ fulfill the assumptions of 2.2.1 with $\dim \mathcal{H} > 1$ and let \mathcal{H}_0 , $\hat{U}_0(a, \Lambda)$, Ω_0 , D_0 , $\hat{\Phi}_0(x)$ be the corresponding objects of the free field theory described in 2.1. If also $\hat{\Phi}(x)$ fulfills the free KLEIN-GORDON equation*

$$(\square + m^2) \hat{\Phi}(x) = 0$$

and if $D = \mathcal{F}(\mathbb{R}^4)\Omega$ then there are a unitary mapping $\hat{V} : \mathcal{H}_0 \longrightarrow \mathcal{H}$ and a constant $\lambda > 0$ with:

$$\begin{aligned} D &= \hat{V} D_0, & \Omega &= \hat{V} \Omega \\ \hat{\Phi}(x) &= \lambda \hat{V} \hat{\Phi}_0(x) \hat{V}^{-1}, \\ \hat{U}(a, \Lambda) &= \hat{V} \hat{U}_0(a, \Lambda) \hat{V}^{-1} \quad \forall (a, \Lambda) \in \mathcal{P}_+^\dagger. \end{aligned}$$

Sketch of proof: Assume that $\hat{\Phi}(x)$ fulfills the KLEIN-GORDON equation. This implies $(\square_\xi + m^2) \mathcal{W}(\xi) = 0$ and hence, by (2.53),

$$\text{supp}\tilde{W} \subset M_m = \{P \in \mathbb{R}^4 : p^0 = \omega_{\mathbf{p}}\}. \quad (2.61)$$

From the proof of Lemma 2.2.10 we also know that there is a positive L_+^\dagger -invariant measure μ with

$$\langle \Omega | \hat{\Phi}(x) \hat{\Phi}(y) \Omega \rangle = \mathcal{W}(x - y) = \int e^{ip(x-y)} \mu(dp)$$

Therefore, there must exist a $\lambda^2 \geq 0$ with

$$\mu(dp) = \frac{\lambda^2}{(2\pi)^3} \theta(p^0) \delta(p^2 - m^2) dp.$$

Without restriction of generality we may assume $\lambda = 1$. Then we have⁵⁴

$$\left\langle \Omega \mid \hat{\Phi}(x)\hat{\Phi}(y)\Omega \right\rangle = i\Delta_m^{(+)}(x-y) \stackrel{\text{def}}{=} (2\pi)^{-3} \int \theta(p^0)\delta(p^2 - m^2)e^{-ip(x-y)} dp. \quad (2.62)$$

The KLEIN-GORDON equation allows us to define

$$\hat{\Phi}^{(\pm)}(x) \stackrel{\text{def}}{=} (2\pi)^{-2} \int_{\pm p^0 \geq 0} \tilde{\hat{\Phi}}(p) e^{-ipx} dp. \quad (2.63)$$

With this definition we have

$$\hat{\Phi}(x) = \hat{\Phi}^{(+)}(x) + \hat{\Phi}^{(-)}(x), \quad (2.64)$$

$$\hat{\Phi}^{(+)}(x)\Omega = 0, \quad (2.65)$$

and hence

$$\left\langle \Omega \mid \hat{\Phi}^{(+)}(x)\hat{\Phi}^{(+)}(y)\Omega \right\rangle = \left\langle \Omega \mid \hat{\Phi}^{(-)}(x)\hat{\Phi}^{(+)}(y)\Omega \right\rangle = \left\langle \Omega \mid \hat{\Phi}^{(-)}(x)\hat{\Phi}^{(-)}(y)\Omega \right\rangle = 0, \quad (2.66)$$

the latter because of $\hat{\Phi}^{(-)}(x)^* = \hat{\Phi}^{(+)}(x)$. Moreover, by (2.62), it is clear that

$$[\hat{\Phi}(x), \hat{\Phi}(y)]_- = \left\langle \Omega \mid [\hat{\Phi}(x), \hat{\Phi}(y)]_- \Omega \right\rangle \hat{1} \wedge D \quad (2.67)$$

would imply

$$[\hat{\Phi}(x), \hat{\Phi}(y)]_- = i\Delta_m(x-y)\hat{1} \wedge D \quad (2.68)$$

(recall (2.47)). Since statements corresponding to (2.63), (2.65), and (2.68) hold for the free theory, this shows that (2.67) would imply that the n -point functions of $\hat{\Phi}(x)$ are the same⁵⁵ as those of $\hat{\Phi}_0(x)$. In view of the GNS representation, described by Theorem 1.3.8 (recall Footnote 57 of Chapter 1), therefore, it is sufficient to prove (2.67). By Corollary 2.2.13, (2.67) follows from

$$[\hat{\Phi}(x), \hat{\Phi}(y)]_- \Omega \sim \Omega. \quad (2.69)$$

To prove (2.69), let us consider states of the form

$$\check{\Psi} = \hat{\Phi}^{(+)}(\varphi_+)\hat{\Phi}^{(-)}(\varphi_-)\Omega, \quad \varphi_{\pm} \in \mathcal{S}(\mathbb{R}^4).$$

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⁵⁴Vice versa, (2.62) implies the KLEIN-GORDON equation, by Corollary 2.2.13. Therefore, assumption (2.61) would have been sufficient. In this sense the results of (Baumann, 1986) are much stronger than the Jost-Schroer theorem. For generalization in a different direction see (Steinmann, 1982).

⁵⁵By (2.64) it is sufficient to check the vacuum expectation values $\left\langle \Omega \mid \hat{\Phi}^{(\sigma_1)}(x_1) \cdots \hat{\Phi}^{(\sigma_n)}(x_n)\Omega \right\rangle$ for arbitrary $n \in \mathbb{N}$ and $\sigma_\nu \in \{+, -\}$. By (2.68), then, it is sufficient to check the expectation values of the form $\left\langle \Omega \mid \hat{\Phi}^{(+)}(x_1) \cdots \hat{\Phi}^{(+)}(x_{n_+})\hat{\Phi}^{(-)}(x_{n_++1}) \cdots \hat{\Phi}^{(-)}(x_{n_++n_-})\Omega \right\rangle$. By (2.65), however, all of them vanish, as for the free field.

For such $\check{\Psi}$ Lemma 2.2.17 implies

$$\check{\Psi} = \hat{E} \left(\overline{V}_+ \cap (-\text{supp}\hat{\Phi}^{(+)} - \text{supp}\hat{\Phi}^{(-)}) \right) \check{\Psi}.$$

Since⁵⁶

$$\overline{V}_+ \cap (-\text{supp}\hat{\Phi}^{(+)} - \text{supp}\hat{\Phi}^{(-)}) = \{0\},$$

this means $\check{\Psi} = \hat{E}(\{0\})\check{\Psi}$. Thus, by (2.58), $\check{\Psi}$ is translational invariant and thus a multiple of Ω , by Borchers' theorem. Therefore:

$$\begin{aligned} \hat{\Phi}^{(+)}(x)\hat{\Phi}^{(-)}(y)\Omega &= \left\langle \Omega \mid \hat{\Phi}^{(+)}(x)\hat{\Phi}^{(-)}(y)\Omega \right\rangle \Omega \\ &\stackrel{(2.64),(2.66)}{=} \left\langle \Omega \mid \hat{\Phi}(x)\hat{\Phi}(y)\Omega \right\rangle \\ &\stackrel{(2.62)}{=} i\Delta_m^{(+)}(x-y)\Omega. \end{aligned}$$

By (2.65), (2.64), and

$$\Delta_m(x-y) = \Delta_m^{(+)}(x-y) - \Delta_m^{(+)}(y-x)$$

this implies

$$[\hat{\Phi}(x), \hat{\Phi}(y)]_{-\Omega} = i\Delta_m(x-y)\Omega + [\hat{\Phi}^{(-)}(x), \hat{\Phi}^{(-)}(y)]_{-\Omega}.$$

Therefore, it is sufficient to prove

$$F_{\Psi}(x, y) \stackrel{\text{def}}{=} \left\langle \Psi \mid [\hat{\Phi}^{(-)}(x), \hat{\Phi}^{(-)}(y)]_{-\Omega} \right\rangle = 0$$

for all $\Psi \in D$. This, however, is guaranteed by Corollary 2.2.9 since, obviously, $\text{supp}F_{\Psi} \neq \mathbb{R}^8$ and $\text{supp}\overline{F}_{\Psi} \subset \overline{V}_+ \times \overline{V}_+$ (recall Footnote 56). ■

2.2.5 PCT Theorem

The defining representation of the **Lorentz group** $L = L(\mathbb{R})$ is well known to be given by the real 4×4 -matrices Λ fulfilling

$$\Lambda^T \eta \Lambda = \eta \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (2.70)$$

Similarly, the defining representation of the **complex Lorentz group** $L = L(\mathbb{C})$ is given by the **complex** 4×4 -matrices Λ fulfilling (2.70). Its subgroup

$$L_+(\mathbb{C}) \stackrel{\text{def}}{=} \{\Lambda \in L(\mathbb{C}) : \det \Lambda = +1\}$$

is called the **proper complex Lorentz group**. Obviously:

$$L_+^\dagger \subset L_+(\mathbb{C}) \ni +\mathbb{1}_4, -\mathbb{1}_4. \quad (2.71)$$

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⁵⁶By the KLEIN-GORDON equation we have $\text{supp}\tilde{\Phi} \subset M_m \cup (-M_m)$ and hence $\text{supp}\widetilde{\hat{\Phi}}^{(\pm)} \subset \pm M_m$, by (2.63).

Theorem 2.2.19 (BARGMANN-HALL-WIGHTMAN) *Let $n, N \in \mathbb{N}$, $n > 1$, let $\Lambda \rightarrow S(\Lambda)$ be an irreducible $N \times N$ -matrix representation of L_+^\uparrow , and let $\mathcal{W}_1, \dots, \mathcal{W}_N$ be holomorphic functions on \mathcal{T}'_{n-1} with*

$$\mathcal{W}_\mu(z_1, \dots, z_{n-1}) = \sum_{\nu=1}^N S_{\mu\nu}(\Lambda^{-1}) \mathcal{W}_\nu(\Lambda z_1, \dots, \Lambda z_{n-1}) \quad \forall \mu \in \{1, \dots, N\} \quad (2.72)$$

for all $(z_1, \dots, z_{n-1}) \in \mathcal{T}_{n-1}$ and all $\Lambda \in L_+^\uparrow$. Then the \mathcal{W}_ν have unique single-valued analytic continuations onto the **extended tube**

$$\mathcal{T}'_{n-1} \stackrel{\text{def}}{=} \{(\Lambda z_1, \dots, \Lambda z_{n-1}) : \Lambda \in L_+(\mathbb{C}), (z_1, \dots, z_{n-1}) \in \mathcal{T}_{n-1}\}$$

fulfilling (2.72) for all $(z_1, \dots, z_{n-1}) \in \mathcal{T}'_{n-1}$ and all $\Lambda \in L_+(\mathbb{C})$, where $S(\Lambda)$ is to be extended to the corresponding irreducible representation of $L_+(\mathbb{C})$.

Proof: See (Bogush and Fedorov, 1977, Sect. 3.4). ■

Corollary 2.2.20 *Let all the assumptions of 2.2.1, with possible exception of microcausality,⁵⁷ be fulfilled and let $1 < n \in \mathbb{N}$. Then the LAPLACE transform $(\mathcal{L}\widetilde{\mathcal{W}})$ of $\widetilde{\mathcal{W}}$ has a single valued analytic continuation onto the extended tube \mathcal{T}'_{n-1} fulfilling the conditions*

$$\begin{aligned} (\mathcal{L}\widetilde{\mathcal{W}})(\Lambda z_1, \dots, \Lambda z_{n-1}) &= (\mathcal{L}\widetilde{\mathcal{W}})(z_1, \dots, z_{n-1}) \\ &\forall \Lambda \in L_+(\mathbb{C}), (z_1, \dots, z_{n-1}) \in \mathcal{T}'_{n-1} \end{aligned} \quad (2.73)$$

and⁵⁸

$$\begin{aligned} \mathcal{W}(\psi) &= \int (\mathcal{L}\widetilde{\mathcal{W}})(\xi_1, \dots, \xi_{n-1}) \psi(\xi_1, \dots, \xi_{n-1}) d\xi_1 \cdots d\xi_{n-1} \\ &\forall \psi \in \mathcal{S}(\mathcal{T}'_{n-1} \cap \mathbb{R}^{4(n-1)}). \end{aligned} \quad (2.74)$$

Sketch of proof: Note, first of all, that (2.52) and (2.51) imply

$$\mathcal{W}(\Lambda \xi_1, \dots, \Lambda \xi_{n-1}) = \mathcal{W}(\xi_1, \dots, \xi_{n-1}) \quad \forall \Lambda \in L_+^\uparrow.$$

This together with (2.53) and Lemma 2.2.7 gives

$$(\mathcal{L}\widetilde{\mathcal{W}})(\xi_1 + i\eta_1, \dots, \xi_{n-1} + i\eta_{n-1}) = (\mathcal{L}\widetilde{\mathcal{W}})(\Lambda \xi_1 + i\Lambda \eta_1, \dots, \Lambda \xi_{n-1} + i\Lambda \eta_{n-1}) \quad \forall \Lambda \in L_+^\uparrow$$

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⁵⁷Microcausality would allow further holomorphic continuation when $n > 2$ (see, e.g., (Tomozawa, 1963)).

⁵⁸Note that $\mathcal{T}'_{n-1} \cap \mathbb{R}^{4(n-1)}$ is an open subset of $\mathbb{R}^{4(n-1)}$, since \mathcal{T}'_{n-1} is open in $\mathbb{C}^{4(n-1)}$.

as an equation for holomorphic functions on \mathcal{T}_{n-1} . Therefore, by Theorem 2.2.19, $(\mathcal{L}\widetilde{\mathcal{W}})$ has a single valued analytic continuation onto the extended tube \mathcal{T}'_{n-1} fulfilling (2.73). From Lemma 2.2.7 we also know that

$$\mathcal{W}(\psi) = \lim_{V \ni \eta_1, \dots, \eta_{n-1} \rightarrow 0} \int (\mathcal{L}\widetilde{\mathcal{W}})(\xi_1 + i\eta_1, \dots, \xi_{n-1} + i\eta_{n-1}) \psi(\xi_1, \dots, \xi_{n-1}) d\xi_1 \cdots d\xi_{n-1}$$

for all $\psi \in \mathcal{S}(\mathbb{R}^{4(n-1)})$. Since every Jost point⁵⁹

$$\check{\xi} \stackrel{\text{def}}{=} (\check{\xi}_1, \dots, \check{\xi}_{n-1}) \in \mathcal{J}_{n-1} \stackrel{\text{def}}{=} \mathcal{T}'_{n-1} \cap \mathbb{R}^{4(n-1)}$$

has a complex open neighborhood $\mathcal{O}_{\check{\xi}} \subset \mathcal{J}_{n-1}$, this implies

$$\mathcal{W}(\psi) = \int (\mathcal{L}\widetilde{\mathcal{W}})(\check{\xi}) \psi(\check{\xi}) d\check{\xi} \quad \forall \psi \in \mathcal{S}(\mathcal{O}_{\check{\xi}} \cap \mathbb{R}^{4(n-1)}).$$

From this (2.74) follows by standard distribution theoretical techniques (choice of a suitable *partition of unity*). ■

Theorem 2.2.21 (Jost) *Let $1 < n \in \mathbb{N}$ and $(\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{4(n-1)}$. Then*

$$\begin{aligned} & (\xi_1, \dots, \xi_{n-1}) \in \mathcal{T}'_{n-1} \\ \iff & \left(\sum_{\nu=1}^{n-1} \lambda_{\nu} \xi_{\nu} \right) \left(\sum_{\nu=1}^{n-1} \lambda_{\nu} \xi_{\nu} \right) < 0 \quad \text{for all } \lambda_1, \dots, \lambda_{n-1} \geq 0 \text{ with } \sum_{\nu=1}^{n-1} \lambda_{\nu} > 0. \end{aligned}$$

Proof: See (Streater and Wightman, 1989, Theorem 2-12). ■

Corollary 2.2.22 (PCT Theorem) *Let all the assumptions of 2.2.1, with possible exception of microcausality, be fulfilled and let $1 < n \in \mathbb{N}$. Then the PCT condition*

$$W(x_1, \dots, x_n) = W(-x_n, \dots, -x_1) \quad (2.75)$$

*is equivalent to the condition of **weak local commutativity***

$$W(x_1, \dots, x_n) = W(x_n, \dots, x_1) \quad \text{for } (x_1 - x_2, \dots, x_{n-1} - x_n) \in \mathcal{J}_{n-1}. \quad (2.76)$$

Proof: By Corollary 2.2.20, since $-\mathbb{1}_4 \in L_+(\mathbb{C})$, (2.76) is equivalent to⁶⁰

$$W(x_1, \dots, x_n) = W(-x_n, \dots, -x_1) \quad \text{for } (x_1 - x_2, \dots, x_{n-1} - x_n) \in \mathcal{J}_{n-1},$$

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⁵⁹The elements of $\mathcal{T}'_{n-1} \cap \mathbb{R}^{4(n-1)}$ are usually called **Jost points**.

⁶⁰Since $\mathcal{J}_1 = \{\xi \in \mathbb{R}^4 : \xi \times 0\}$, this shows once more that

$$\langle \Omega | [\hat{\Phi}(x), \hat{\Phi}(y)]_- \Omega \rangle = 0 \quad \text{for } x \times y$$

holds even if microcausality is not assumed, as a consequence of the other WIGHTMAN axioms.

i.e. to

$$F(\check{\xi}) \stackrel{\text{def}}{=} \mathcal{W}(\xi_1, \dots, \xi_{n-1}) - \mathcal{W}(\xi_{n-1}, \dots, \xi_1) = 0 \quad \forall \check{\xi} \in \mathcal{J}_{n-1}. \quad (2.77)$$

By Corollary 2.2.9, since (2.53) implies

$$\text{supp } F \subset \overline{V_+} \times \dots \times \overline{V_+},$$

(2.77) and hence (2.76) is equivalent to $F = 0$, i.e. to (2.75). ■

Exercise 39 Show that (2.75) is equivalent to existence of an anti-unitary Operator $\hat{\theta}$ fulfilling the conditions

$$\begin{aligned} \hat{\theta}\hat{\theta} &= \hat{1}, \quad \hat{\theta}\Omega = \Omega, \\ \hat{\theta}\hat{\Phi}(\varphi)\hat{\theta} &= \left(\int \hat{\Phi}(-x)\varphi(x) dx \right)^* \wedge D \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^4), \end{aligned}$$

and

$$\hat{\theta}\hat{U}(a)\hat{\theta} = \hat{U}(-a) \quad \forall a \in \mathbb{R}^4.$$

2.3 S-Matrix for Self-interacting Neutral Scalar Fields

2.3.1 General Scattering Theory

The main concern of scattering theory is asymptotic (for $t \rightarrow \pm\infty$) identification of an interacting system (IS) with a suitable “free” system (FS):

$$\begin{aligned} \check{\mathcal{H}} &\stackrel{\text{def}}{=} \text{set of all (pure) states of the IS}, \\ \check{\mathcal{H}}_0 &\stackrel{\text{def}}{=} \text{set of all (pure) states of the FS}. \end{aligned}$$

Here, we use the HEISENBERG *picture*, i.e. the states describe the corresponding system for all times until some “measurement” is taking place. Whether the considered systems are classical or quantum does not matter, so far. Thus, given $\check{\Psi} \in \check{\mathcal{H}}$, the basic problem is to find the (hopefully unique) “free” states $\check{\Psi}_{\pm} \in \check{\mathcal{H}}_0$ such that

$$\text{for } t \rightarrow \pm\infty \quad \check{\Psi} \text{ “looks like” } \check{\Psi}_{\pm}.$$

The precise meaning of the latter has to be specified by some *asymptotic condition* (AC) as sketched in Fig. 2.4. For example, in **potential scattering of classical particles**, sketched in Fig. 2.5, the Heisenberg states are given by the solutions $\mathbf{x}(t)$ of the classical equations of motion and $\mathbf{x}_{\pm}(t)$ being *free* means:

$$\mathbf{x}_{\pm}(t) = \mathbf{x}_{\pm} + \mathbf{v}_{\pm}t.$$

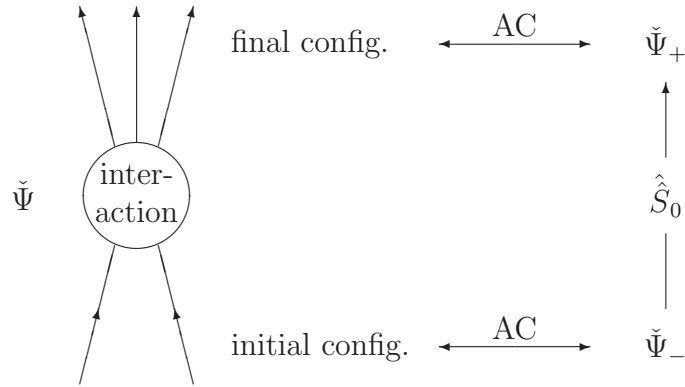


Figure 2.4: Asymptotic identification of the IS (left) with the FS (right) via the AC

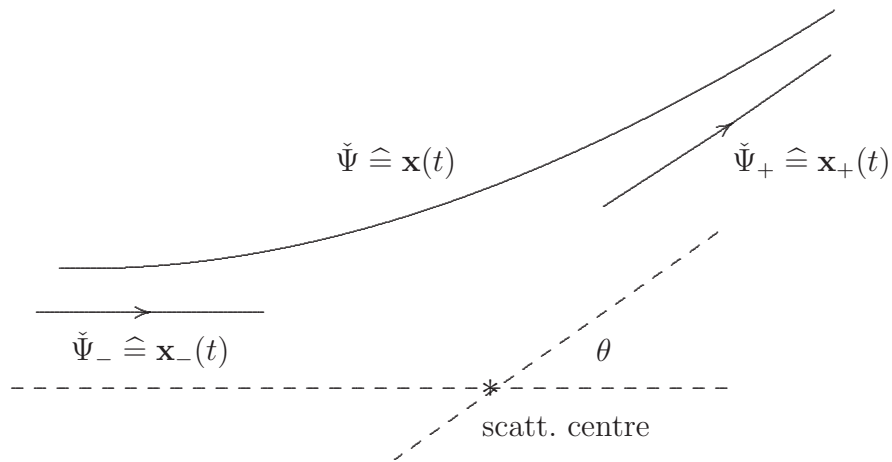


Figure 2.5: Asymptotics for a classical particle

For potentials of sufficiently short range forces the *asymptotic condition* is:⁶¹

$$\mathbf{v}_{\pm} = \lim_{t \rightarrow \pm\infty} \dot{\mathbf{x}}(t), \quad \mathbf{x}_{\pm} = \lim_{t \rightarrow \pm\infty} \mathbf{x}(t) - \mathbf{v}_{\pm} t.$$

$\check{\Psi} \in \check{\mathcal{H}}$ is called a *scattering state*, if there are $\check{\Psi}_{-} \in \check{\mathcal{H}}_0$ and $\check{\Psi}_{+} \in \check{\mathcal{H}}_0$ fulfilling the AS for $\check{\Psi}$. Obviously, this condition need not always be fulfilled (*bounded states*, particle *capture* etc.).

FS and AC have to meet the following **requirement**:

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⁶¹See (Reed und Simon, 1972, Vol. III, Thm. XI.1). For the Coulomb potential the “free states” have to be taken of the form

$$\mathbf{x}_{\pm}(t) = \mathbf{x}_{\pm} + \mathbf{v}_{\pm} t + \mathbf{d}_{\pm} \ln t$$

(see (Reed und Simon, 1972, Vol. III, Sect. 9)).

For every $\check{\Phi}_0 \in \check{\mathcal{H}}_0$ there is exactly one $\check{\Psi} \in \check{\mathcal{H}}$ with $\check{\Psi}_- = \check{\Phi}_0$; and similarly for ‘+’ instead of ‘-’.

Then we can define the following **generalized wave operators**:

$$\begin{aligned} \hat{V}_{\text{in}}\check{\Phi}_0 &\stackrel{\text{def}}{=} \text{the state } \check{\Psi} \in \check{\mathcal{H}} \text{ for which } \check{\Psi}_- = \check{\Phi}_0, \\ \hat{V}_{\text{out}}\check{\Phi}_0 &\stackrel{\text{def}}{=} \text{the state } \check{\Psi}' \in \check{\mathcal{H}} \text{ for which } \check{\Psi}'_+ = \check{\Phi}_0. \end{aligned} \quad (2.78)$$

Now, the subject of scattering theory is to study the relation between $\check{\Psi}_- = \hat{V}_{\text{in}}^{-1}\check{\Psi}$ and $\check{\Psi}_+ = \hat{V}_{\text{out}}^{-1}\check{\Psi}$ for arbitrary scattering states $\check{\Psi}$.

The **scattering operator in the HEISENBERG picture** is

$$\hat{S} \stackrel{\text{def}}{=} \hat{V}_{\text{in}}\hat{V}_{\text{out}}^{-1} \quad (2.79)$$

and maps $\check{\mathcal{H}}_{\text{out}} \stackrel{\text{def}}{=} \hat{V}_{\text{out}}\check{\mathcal{H}}_0$ one-to-one onto $\check{\mathcal{H}}_{\text{in}} \stackrel{\text{def}}{=} \hat{V}_{\text{in}}\check{\mathcal{H}}_0$:

$$\hat{S} \underbrace{\hat{V}_{\text{out}}\check{\Phi}_0}_{\text{looks like } \check{\Phi}_0 \text{ for } t \rightarrow +\infty} = \underbrace{\hat{V}_{\text{in}}\check{\Phi}_0}_{\text{looks like } \check{\Phi}_0 \text{ for } t \rightarrow -\infty}. \quad (2.80)$$

Definition (2.79) has the advantage of being applicable even in case $\check{\mathcal{H}}_{\text{in}} \not\subset \check{\mathcal{H}}_{\text{out}}$, contrary to definition (2.82) below, and of being independent of the special choice for the realization of the free system. Its drawback is that \hat{S} describes the relation between $\check{\Psi}_-$ and $\check{\Psi}_+$ only indirectly (via \hat{V}_{out} or \hat{V}_{in}):

$$\left. \begin{aligned} \hat{S}\hat{V}_{\text{out}}\check{\Psi}_- &\left(\begin{array}{l} = \hat{V}_{\text{in}}\check{\Psi}_- = \check{\Psi} \\ \text{(2.79)} \end{array} \right) = \hat{V}_{\text{out}}\check{\Psi}_+ \\ \hat{S}\hat{V}_{\text{in}}\check{\Psi}_- &\left(\begin{array}{l} = \hat{S}\check{\Psi} = \hat{S}\hat{V}_{\text{out}}\check{\Psi}_+ \\ \text{(2.79)} \end{array} \right) = \hat{V}_{\text{in}}\check{\Psi}_+ \end{aligned} \right\} \text{for scattering states } \check{\Psi}. \quad (2.81)$$

If $\check{\mathcal{H}}_{\text{in}} \subset \check{\mathcal{H}}_{\text{out}}$, this relation may be described directly via the **scattering operator in the interaction picture**

$$\hat{S}_0 \stackrel{\text{def}}{=} \hat{V}_{\text{out}}^{-1}\hat{V}_{\text{in}}, \quad (2.82)$$

namely:

$$\hat{S}_0\check{\Psi}_- = \check{\Psi}_+. \quad (2.83)$$

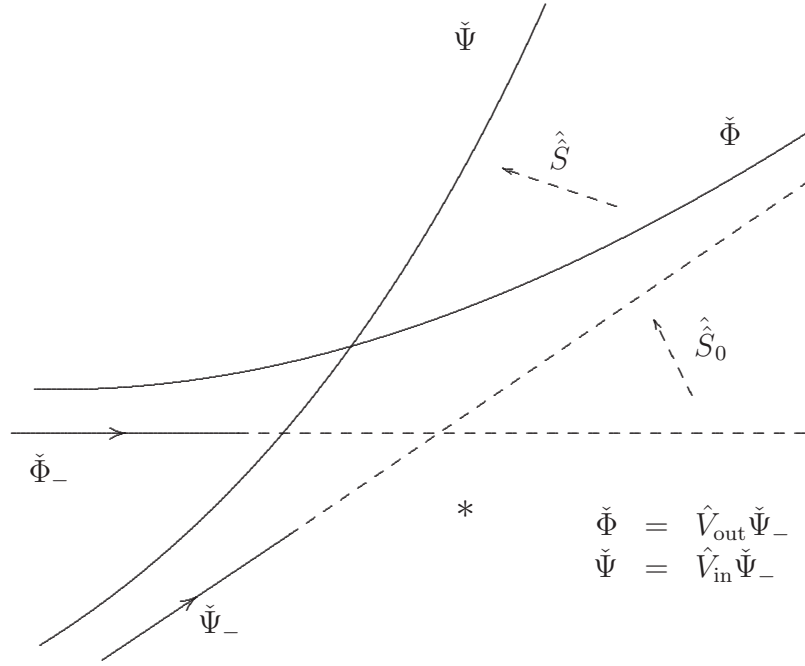
In this case, i.e. when $\check{\mathcal{H}}_{\text{in}}$ contains only scattering states (no capture),

$$\hat{S} = \hat{V}_{\text{out}}\hat{S}_0\hat{V}_{\text{out}}^{-1}. \quad (2.84)$$

In case of **weak asymptotic completeness**, i.e. if $\check{\mathcal{H}}_{\text{in}} = \check{\mathcal{H}}_{\text{out}}$, we also have

$$\hat{S} = \hat{V}_{\text{in}}\hat{S}_0\hat{V}_{\text{in}}^{-1}, \quad (2.85)$$

since then $D_{\hat{V}_{\text{in}}^{-1}} = D_{\hat{V}_{\text{out}}^{-1}}$. For potential scattering of classical particles the action of \hat{S} and \hat{S}_0 is sketched in Fig. 2.6.

Figure 2.6: Two versions of the S -matrix

2.3.2 Asymptotic condition for massive neutral scalar particles

In quantum mechanics without superselection rules, the pure states of the WS resp. FS are given by the 1-dimensional subspaces $\check{\Psi}$ resp. $\check{\Psi}_0$ of some complex HILBERT space \mathcal{H} resp. \mathcal{H}_0 :

$$\check{\mathcal{H}}_{(0)} = \{ \check{\Psi}_{(0)} = \{ \lambda \Psi_{(0)} : \lambda \in \mathbb{C} \} : \Psi_{(0)} \in \mathcal{H}_{(0)} \} .$$

The generalized wave operators $\hat{V}_{(\text{in})}^{\text{out}}$ are given by isometric (linear) mappings

$$\hat{V}_{(\text{in})}^{\text{out}} : \mathcal{H}_0 \longrightarrow \mathcal{H}_{(\text{in})}^{\text{out}} \stackrel{\text{def}}{=} \hat{V}_{(\text{in})}^{\text{out}} \mathcal{H}_0 \subset \mathcal{H}$$

via:

$$\hat{V}_{(\text{in})}^{\text{out}} \check{\Psi}_0 \stackrel{\text{def}}{=} \left\{ \lambda \hat{V}_{(\text{in})}^{\text{out}} \Psi_0 : \lambda \in \mathbb{C} \right\} \quad \text{for } \Psi_0 \in \mathcal{H}_0 .$$

Let us consider the theory of a neutral scalar field as specified in 2.2.1, describing the IS, and assume:

1. The restriction of the representation $\hat{U}(a, \Lambda)$ of \mathcal{P}_+^\uparrow to $\hat{E}(M_m)\mathcal{H}$ is irreducible.
2. The free field theory described in 2.1.3 represents a FS suitable for asymptotic description of the IS.

3. The corresponding scattering theory fulfills the condition of **asymptotic completeness**:

$$\mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}} = \mathcal{H}.$$

By asymptotic completeness, then, the scattering isometries $\hat{V}_{(\text{in})}^{\text{out}}$ are even unitary mappings from \mathcal{H}_0 onto \mathcal{H} . The chosen FS can only be suitable if

$$\hat{U}(a, \Lambda) \hat{V}_{(\text{in})}^{\text{out}} = \hat{V}_{(\text{in})}^{\text{out}} \hat{U}_0(a, \Lambda) \quad \forall (a, \Lambda) \in \mathcal{P}_+^\uparrow. \quad (2.86)$$

This implies

$$\text{supp} \hat{E} = \text{supp} \hat{E}_0 = \{0\} \cup M_m \cup \{p \in \mathbb{R}^4 : p^2 \geq (2m)^2, p^0 > 0\} \quad (2.87)$$

and

$$\hat{V}_{(\text{in})}^{\text{out}} \mathcal{H}_0^{(0)} = \hat{E}(\{0\}) \mathcal{H}, \quad \hat{V}_{(\text{in})}^{\text{out}} \mathcal{H}_0^{(1)} = \hat{E}(M_m) \mathcal{H}.$$

Since $\hat{U}(a, \Lambda) \wedge \hat{E}(M_m) \mathcal{H}$ is irreducible and since the vacuum state is uniquely characterized by \mathcal{P}_+^\uparrow -invariance, we may assume without loss of generality:

$$\mathcal{H}_0^{(0)} \oplus \mathcal{H}_0^{(1)} = \hat{E}(\{0\} \cup M_m) \mathcal{H}, \quad \hat{V}_{(\text{in})}^{\text{out}} \wedge (\mathcal{H}_0^{(0)} \oplus \mathcal{H}_0^{(1)}) = \hat{1} \wedge (\mathcal{H}_0^{(0)} \oplus \mathcal{H}_0^{(1)}). \quad (2.88)$$

By unitarity of $\hat{V}_{(\text{in})}^{\text{out}}$, then, it is sufficient to determine the vectors

$$\Psi_{(\text{in})}^{\text{out}}(\check{\chi}_1, \dots, \check{\chi}_n) \stackrel{\text{def}}{=} \hat{V}_{(\text{in})}^{\text{out}}(\hat{a}_0^*(\check{\chi}_1) \cdots \hat{a}_0^*(\check{\chi}_n) \Omega) \quad (2.89)$$

(recall (2.33)) for all $n > 1$ and functions $\check{\chi}_1, \dots, \check{\chi}_n \in \mathcal{D}(\mathbb{R}^3)$ that are **nonoverlapping**, i.e.:

$$\nu \neq \mu \implies \text{supp} \check{\chi}_\nu \cap \text{supp} \check{\chi}_\mu = \emptyset.$$

We want to characterize the states corresponding to vectors of the form (2.89) by their expectation values for localized measurements corresponding to **bounded** observables.⁶² The use of the field $\hat{\phi}(x)$ for this is just to associate with every open region $\mathcal{O} \subset \mathbb{R}^4$ the corresponding **algebra of local observables**⁶³ $\mathcal{A}(\mathcal{O})$, i.e. the VON NEUMANN algebra generated by all **bounded** observables corresponding to measurements performable within \mathcal{O} . Once the local algebras $\mathcal{A}(\mathcal{O})$ are specified, we may forget about the field $\hat{\phi}(x)$ as far as the S matrix is concerned.

In order to be able to interpret the closed smeared field operators $\int dx \hat{\Phi}(x) \varphi(x)$, where $\varphi = \bar{\varphi} \in \mathcal{S}(\mathbb{R}^4)$, as observables of the φ -weighted ‘field strength’ $\int dx \Phi(x) \varphi(x)$, let us assume that before closure the operators $\hat{\Phi}(\varphi) \stackrel{\text{def}}{=} \int dx \hat{\Phi}(x) \varphi(x)$ on D are

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⁶²A more general formalism, suitable also for nonlocalizable fields, was developed in (Lücke, 1983).

⁶³See (Thomas and Wichmann, 1998) and references given there for further details.

essentially self-adjoint and that in case $\text{supp } \varphi_1 \not\propto \text{supp } \varphi_2$ also the spectral projection operators of $\hat{\Phi}(\varphi_1)$ commute⁶⁴ with those of $\hat{\Phi}(\varphi_2)$. Then,⁶⁵ given an open region $\mathcal{O} \subset \mathbb{R}^4$, $\mathcal{A}(\mathcal{O})$ should be identified with the VON NEUMANN subalgebra of $\mathcal{L}(\mathcal{H})$ generated by all spectral operators of the selfadjoint operators $\hat{\Phi}(\varphi)$, $\varphi = \bar{\varphi} \in \mathcal{S}(\mathcal{O})$. This also ensures that the following four conditions are fulfilled:⁶⁶

$$\mathcal{O}_1 \subset \mathcal{O}_2 \implies \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2), \quad (2.90)$$

$$\mathcal{O}_1 \times \mathcal{O}_2 \implies \left[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2) \right]_- = \{\hat{0}\}, \quad (2.91)$$

$$\hat{U}(a, \Lambda) \mathcal{A}(\mathcal{O}) \hat{U}(a, \Lambda)^{-1} = \mathcal{A}(\Lambda \mathcal{O} + a), \quad (2.92)$$

$$\left(\bigcup_{R>0} \mathcal{A}(U_R(0)) \right)' = \{\lambda \hat{1} : \lambda \in \mathbb{C}\}. \quad (2.93)$$

By RUELLE's lemma (Lemma 2.1.1) the influence of an asymptotic particle on measurements outside its velocity cone should fall off very rapidly. This statement has to be made precise in order to get a suitable asymptotic condition:

Asymptotic Condition:⁶⁷ Let $\check{\chi}_1, \dots, \check{\chi}_n \in \mathcal{D}(\mathbb{R}^3)$ be **nonoverlapping** and let the vectors $\hat{a}_0^*(\chi_1)\Omega_0, \dots, \hat{a}_0^*(\chi_n)\Omega_0 \in \mathcal{H}_0^{(1)}$ be normalized. Moreover, let $\check{K} = -\check{K}$ be a closed cone for which

$$\check{K} \cap K_{\check{\chi}_n} = \{0\}, \quad K_{\check{\chi}_n} \stackrel{\text{def}}{=} \left\{ \left(t, \frac{\mathbf{p}}{\omega_{\mathbf{p}}} t \right) : t \in \mathbb{R}, \mathbf{p} \in \text{supp } \check{\chi}_n \right\}.$$

Then, for sufficiently small⁶⁸ $\epsilon > 0$ there exists a sequence of positive numbers C_1, C_2, \dots for which

$$\begin{aligned} & |t|^N \left| \left\langle \Psi_{(\text{in})}^{\text{out}}(\check{\chi}_1, \dots, \check{\chi}_n) \middle| \hat{A}_t \Psi_{(\text{in})}^{\text{out}}(\check{\chi}_1, \dots, \check{\chi}_n) \right\rangle \right. \\ & \quad \left. - \left\langle \Psi_{(\text{in})}^{\text{out}}(\check{\chi}_1, \dots, \check{\chi}_{n-1}) \middle| \hat{A}_t \Psi_{(\text{in})}^{\text{out}}(\check{\chi}_1, \dots, \check{\chi}_{n-1}) \right\rangle \right| \\ & \leq \|\hat{A}_t\| C_N \quad \forall t > 0, N \in \mathbb{N}, \hat{A}_t \in \mathcal{A}(\check{K} \cap U_{\epsilon|t|}(\Sigma_t)), \end{aligned}$$

where:

$$\Sigma_t \stackrel{\text{def}}{=} \{X \in \mathbb{R}^4 : x^0 = t\}.$$

⁶⁴This is a stronger version of the WIGHTMAN axiom (v); for an interesting sufficient condition see (Borchers and Zimmermann, 1964), again.

⁶⁵For more general considerations concerning the connection between local algebras of bounded operators and local WIGHTMAN fields see (Wollenberg, 1985) and (Driessler et al., 1986b).

⁶⁶Statement (2.93) is a simple consequence of Corollary 2.2.16 and the spectral theorem.

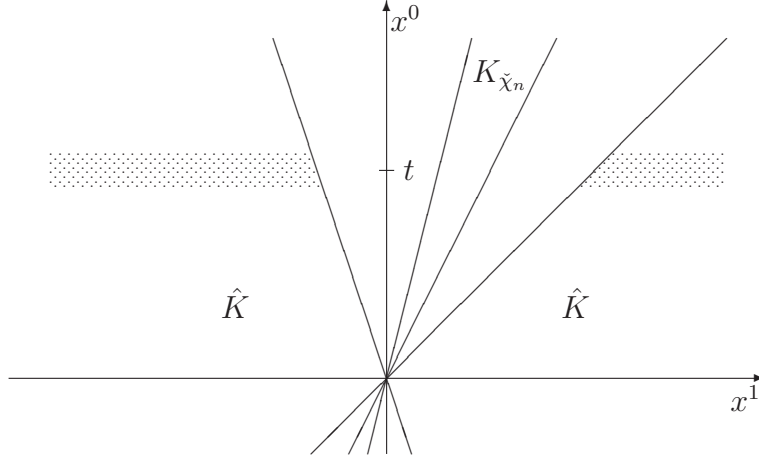


Figure 2.7: The regions (dotted) of measurement $\tilde{K} \cap U_{ct}(\Sigma_t)$, $t > 0$

2.3.3 Evaluation of the Asymptotic Condition

The evaluation of the asymptotic condition depends crucially on the following representation of 1-particle states:

Definition 2.3.1 Given $\tilde{\chi} \in \mathcal{D}(\mathbb{R}^3)$, a sequence $\{\hat{B}_t\}_{t \in \mathbb{R}} \subset \mathcal{L}(\mathcal{H})$ is called a **Haag-Ruelle-Kastler sequence** (HRK sequence) for

$$\Psi = \hat{a}_0^*(\tilde{\chi})\Omega_0 \in \mathcal{H}_0^{(1)} \subset \mathcal{H}$$

if the following three conditions are fulfilled:

(i)

$$\lim_{t \rightarrow \pm\infty} |t|^N \left\| \hat{B}_t \Omega_0 - \Psi \right\| = 0 \quad \forall N \in \mathbb{N}.$$

(ii) For every $N \in \mathbb{N}$ there is a sequence of local operators

$$\hat{A}_t \in \mathcal{A} \left(U_{\frac{|t|}{N}}(K_{\tilde{\chi}} \cap \Sigma_t) \right)$$

with

$$\lim_{t \rightarrow \pm\infty} |t|^N \left\| \hat{A}_t - \hat{B}_t \right\| = 0.$$

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⁶⁷The factor $|t|^N$ with **arbitrary** N is appropriate for short-range forces, only. For long-range forces there are only very limited results (Buchholz, 1977).

⁶⁸The **causal completion** $\mathcal{O}'_t \stackrel{\text{def}}{=} \{x \in \mathbb{R}^4 : x \times \mathcal{O}_t\}$ of $\mathcal{O}_t = \tilde{K} \cap U_{c|t|}(\Sigma_t)$ must not intersect $K_{\tilde{\chi}_n}$.

(iii)

$$\lim_{t \rightarrow \pm\infty} |t|^{-N} \left\| \hat{B}_t \right\| = 0 \quad \text{for sufficiently large } N \in \mathbb{N}.$$

In order to prove existence of HRK sequences with the additional properties formulated in Lemma 2.3.3, below, we need the following variant of Lemma 2.2.17:

Lemma 2.3.2 *Let $\hat{A} \in \mathcal{L}(\mathcal{H})$ and let $\varphi \in \mathcal{S}(\mathbb{R}^4)$. Then for every BOREL $B \subset \mathbb{R}^4$:*

$$\Psi \in \hat{E}(B)\mathcal{H} \implies \int \hat{A}(x) \varphi(x) dx \Psi \in \hat{E}(\overline{V}_+ \cap (B + \text{supp}\tilde{\varphi}))\mathcal{H},$$

where

$$\hat{A}(x) \stackrel{\text{def}}{=} \hat{U}(x) \hat{A} \hat{U}(x)^{-1} \quad \text{for } x \in \mathbb{R}^4.$$

Sketch of proof: Let B and \hat{B} be arbitrary BOREL subsets of \mathbb{R}^4 and let $\Psi \in \hat{E}(B)\mathcal{H}$. Then, since (2.60) implies

$$\hat{E}(B) = (2\pi)^{-2} \int \hat{U}(a) \tilde{\chi}_B(-a) da,$$

where χ_B denotes the characteristic function of B :

$$\begin{aligned} & (2\pi)^4 \hat{E}(\hat{B}) \int \hat{A}(x) \varphi(x) dx \Psi \\ &= \int \hat{U}(\hat{a}) \hat{A}(x) \hat{U}(a) \tilde{\chi}_{\hat{B}}(-\hat{a}) \tilde{\chi}_B(-a) \varphi(x) d\hat{a} da dx \Psi \\ &= \int \hat{A}(x + \hat{a}) \hat{U}(\hat{a} + a) \tilde{\chi}_{\hat{B}}(-\hat{a}) \tilde{\chi}_B(-a) \varphi(x) d\hat{a} da dx \Psi \\ &= \int \hat{A}(x) \hat{U}(a) \tilde{\chi}_{\hat{B}}(-\hat{a}) \tilde{\chi}_B(\hat{a} - a) \varphi(x - \hat{a}) d\hat{a} da dx \Psi. \end{aligned}$$

Since

$$\int \tilde{\chi}_{\hat{B}}(-\hat{a}) \tilde{\chi}_B(\hat{a} - a) \varphi(x - \hat{a}) d\hat{a} = 0 \quad \text{for } \hat{B} \cap (B + \text{supp}\tilde{\varphi}) = \emptyset,$$

this implies

$$\int \hat{A}(x) \varphi(x) dx \Psi \in \hat{E}(B + \text{supp}\tilde{\varphi})\mathcal{H}$$

and hence, by the spectrum condition, the statement of the lemma. ■

Lemma 2.3.3 *For every $\check{\chi} \in \mathcal{D}(\mathbb{R}^3)$ and $\epsilon > 0$ there is a HKR sequence $\{\hat{B}_t\}_{t \in \mathbb{R}}$ for $\hat{a}_0^*(\check{\chi})\Omega_0$ fulfilling the following two conditions.⁶⁹*

(i)

$$\lim_{t \rightarrow \pm\infty} |t|^N \left\| \hat{B}_t^* \hat{B}_t \Omega_0 - \left\langle \Omega_0 \mid \hat{B}_t^* \hat{B}_t \Omega_0 \right\rangle \Omega_0 \right\| = 0 \quad \forall N \in \mathbb{N}.$$

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⁶⁹Actually, by Lemma 2.3.2 and Borchers' theorem, (i) is a consequence of (ii) for sufficiently small $\epsilon > 0$.

(ii)

$$\hat{B}_t^* \hat{E}(B) \mathcal{H} \subset \hat{E}\left(B - U_\epsilon(M_{\check{\chi}})\right) \mathcal{H} \quad \forall t \in \mathbb{R}, \text{BOREL } B \subset \mathbb{R}^4.$$

where

$$M_{\check{\chi}} \stackrel{\text{def}}{=} \{(\omega_{\mathbf{p}}, \mathbf{p}) : \mathbf{p} \in \text{supp}\check{\chi}\}.$$

Sketch of proof: Let $\check{\chi} \in \mathcal{D}(\mathbb{R}^4)$ and $\epsilon > 0$. The essential point is to show that there is an **almost local** operator \hat{A} , i.e. a bounded operator \hat{A} with

$$\lim_{0 < R \rightarrow \infty} R^N \inf \left\{ \left\| \hat{A} - \hat{B} \right\| : \hat{B} \in \mathcal{A}(U_R(0)) \right\} = 0 \quad \forall N \in \mathbb{N},$$

for which

$$\check{f}_{\hat{A}}(\mathbf{p}) = 1 \quad \forall \mathbf{p} \in \text{supp}\check{\chi}, \quad (2.94)$$

where

$$\check{f}_{\hat{A}} \stackrel{\text{def}}{=} \left(\hat{E}(M_m) \hat{A} \Omega_0 \right)_1.$$

By (2.93), $\{0\}$ and \mathcal{H} itself are the only closed subspaces of \mathcal{H} which are invariant with respect to⁷⁰

$$\mathcal{A}_{\text{loc}} \stackrel{\text{def}}{=} \bigcap_{R < 0} \mathcal{A}(U_R(0)).$$

Therefore

$$\overline{\mathcal{A}_{\text{loc}} \Omega_0} = \mathcal{H}$$

and consequently

$$\check{f}_{\hat{A}} \neq 0 \quad \text{for some } \hat{A} \in \mathcal{A}_{\text{loc}}.$$

Moreover, from

$$\begin{aligned} \hat{E}(M_m) \hat{U}(0, \Lambda) \hat{A} \hat{U}(0, \Lambda)^{-1} \Omega_0 &= \hat{U}(0, \Lambda) \hat{E}(M_m) \hat{A} \Omega_0 \\ &= \hat{U}_0(0, \Lambda) \hat{E}(M_m) \hat{A} \Omega_0 \end{aligned} \quad (2.86), (2.88)$$

and (2.4) we conclude

$$\check{f}_{\hat{U}(0, \Lambda) \hat{A} \hat{U}(0, \Lambda)^{-1}}(\mathbf{p}) = \check{f}_{\hat{A}} \left(\overrightarrow{\Lambda^{-1} p} \right)_{|p^0 = \omega_{\mathbf{p}}} \quad \forall \hat{A} \in \mathcal{A}_{\text{loc}}, \mathbf{p} \in \mathbb{R}^3. \quad (2.95)$$

 Therefore, choosing some Haar measure μ on L_+^\uparrow , we get

$$\check{f}_{\hat{A}_{\check{\delta}}}(\mathbf{p}) = \int \check{f}_{\hat{A}} \left(\overrightarrow{\Lambda^{-1} p} \right)_{|p^0 = \omega_{\mathbf{p}}} \check{\delta}(\Lambda) \mu(d\Lambda),$$

 for sufficiently well-behaved $\check{\delta}$, where

$$\hat{A}_{\check{\delta}} \stackrel{\text{def}}{=} \int \hat{U}(0, \Lambda) \hat{A} \hat{U}(0, \Lambda)^{-1} \check{\delta}(\Lambda) \mu(d\Lambda).$$

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⁷⁰The orthogonal projection onto a nontrivial invariant subspace would commute with all elements of \mathcal{A}_{loc} , in contradiction to (2.93).

Now, for suitable $\check{\delta}$

$$\begin{aligned} g(\Lambda') &\stackrel{\text{def}}{=} \check{f}_{\hat{A}\check{\delta}}\left(\overrightarrow{\Lambda'(m, 0)}\right) \\ &= \int \check{f}_{\hat{A}}\left(\overrightarrow{\Lambda^{-1}(m, 0)}\right) \check{\delta}(\Lambda' \Lambda) \mu(d\Lambda) \\ &\in C^\infty(\mathbb{R}^3) \setminus \{0\} \end{aligned}$$

and therefore

$$\check{f}_{\hat{A}} \in C^\infty(\mathbb{R}^3) \setminus \{0\} \quad \text{for suitable } \hat{A} \in \mathcal{A}_{\text{loc}}.$$

By (2.4), we also have

$$\check{f}_{\hat{U}(x)\hat{A}\hat{U}(x)^{-1}}(\mathbf{p}) = \left(e^{ipx} \check{f}_{\hat{A}}(\mathbf{p}) \right) \Big|_{p^0=\omega_{\mathbf{p}}}$$

and hence

$$\check{f}_{\int \hat{U}(x)\hat{A}\hat{U}(x)^{-1}\varphi(x) dx}(\mathbf{p}) = (2\pi)^2 \check{f}_{\hat{A}}(\mathbf{p}) \tilde{\varphi}(\omega_{\mathbf{p}}, \mathbf{p}). \quad (2.96)$$

Therefore

$$0 \leq \check{f}_{\hat{A}} \in C^\infty(\mathbb{R}^3) \setminus \{0\} \quad \text{for suitable almost local } \hat{A}.$$

From this, using first (2.95) and then (2.96) again, we easily get (2.94) for some almost local \hat{A} .

Note that for $\varphi \in \mathcal{S}(\mathbb{R}^4)$ and

$$f^+(x) \stackrel{\text{def}}{=} (2\pi)^{-3/2} \int_{p^0=\omega_{\mathbf{p}}} \check{\chi}(\mathbf{p}) e^{-ipx} d\mathbf{p} \quad (2.97)$$

we have

$$\begin{aligned} \varphi_t(x) &\stackrel{\text{def}}{=} \int_{y^0=t} \varphi(x-y) f^+(y) dy \\ &= (2\pi)^{-2} \int \left((2\pi)^3 \tilde{\varphi}(p) e^{i(\omega_{\mathbf{p}}-p^0)t} \check{\chi}(\mathbf{p}) \right) e^{-ipx} dp. \end{aligned} \quad (2.98)$$

By (2.96) this implies

$$\begin{aligned} \check{f}_{\int \hat{U}(x)\hat{A}\hat{U}(x)^{-1}\varphi_t(x) dx}(\mathbf{p}) &= 2\omega_{\mathbf{p}} \check{\chi}(\mathbf{p}) \\ &= (\hat{a}_0^*(\check{\chi})\Omega_0)_1(\mathbf{p}) \end{aligned}$$

for all $t \in \mathbb{R}$ if

$$\check{f}_{\hat{A}}(\mathbf{p}) = 1 \quad \forall \mathbf{p} \in \text{supp} \check{\chi}$$

and

$$\tilde{\varphi}(\omega_{\mathbf{p}}, \mathbf{p}) = (2\pi)^{-7/2} 2\omega_{\mathbf{p}} \quad \forall \mathbf{p} \in \text{supp} \check{\chi}. \quad (2.99)$$

In other words:

There is an almost local operator \hat{A} with

$$(2.97)-(2.99) \implies \hat{E}(M_m) \int \hat{U}(x)\hat{A}\hat{U}(x)^{-1}\varphi_t(x) dx \Omega_0 = \hat{a}_0^*(\check{\chi})\Omega_0.$$

Choosing φ such that

$$\text{supp}\tilde{\varphi} \cap \text{supp}\hat{E} \subset M_m \quad (2.100)$$

we even have

$$\int \hat{U}(x)\hat{A}\hat{U}(x)^{-1}\varphi_t(x) dx \Omega_0 = \hat{a}_0^*(\tilde{\chi})\Omega_0 \quad \forall t \in \mathbb{R},$$

by Lemma 2.3.2. Since, obviously,

$$\sup_{x \in \mathbb{R}^4} |f^+(x)| < \infty$$

this together with Ruelle's Lemma (Lemma 2.1.1) shows that (2.97) – (2.100) guarantee $\left\{ \hat{B}_t \stackrel{\text{def}}{=} \int_{t \in \mathbb{R}} \hat{U}(x)\hat{A}\hat{U}(x)^{-1}\varphi_t(x) dx \right\}$ to be a HRK sequence for $\hat{a}_0^*(\tilde{\chi})\Omega_0$. In order to fulfill also conditions (i) and (ii) of the lemma, it is sufficient – thanks to Lemma 2.3.2, spectrum condition and Borchers' theorem – to choose φ such that also

$$\text{supp}\tilde{\varphi} \subset U_\delta(M_{\tilde{\chi}})$$

holds with $\delta > 0$ sufficiently small to ensure

$$U_\delta(M_{\tilde{\chi}}) \cap U_\delta(-M_{\tilde{\chi}}) \subset U_m(0). \quad \blacksquare$$

Exercise 40 Show that the HRK sequence of Lemma 2.3.3 may be constructed in the form

$$\hat{B}_t = \int_{x^0=t} \hat{B}(x) i\overleftrightarrow{\partial}_0 f^+(x) dx, \quad \hat{B}(x) \stackrel{\text{def}}{=} \hat{U}(x) \hat{B} \hat{U}(x)^{-1},$$

with f^+ defined by (2.97), where \hat{B} is an almost local operator fulfilling

$$(\square_x + m^2)\hat{B}(x)\Omega_0 = 0$$

and hence

$$\hat{B}_\Sigma \Omega_0 = \hat{a}_0^*(\tilde{\chi})\Omega_0, \quad \hat{B}_\Sigma \stackrel{\text{def}}{=} \int_\Sigma \hat{B}(x) i\overleftrightarrow{\partial}_\mu \chi^+(x) d\sigma_\mu,$$

for every (sufficiently well-behaved) spacelike hypersurface Σ (without finite boundary points). Show that \hat{B} may be chosen such that also

$$\hat{B}_\Sigma^* \hat{B}_\Sigma \Omega_0 \sim \Omega_0$$

holds for all these surfaces.

Theorem 2.3.4 *Let $\check{\chi}_1, \dots, \check{\chi}_n \in \mathcal{D}(\mathbb{R}^3)$ be nonoverlapping and let*

$$\|\hat{a}_0^*(\check{\chi}_\nu)\Omega_0\| = 1 \quad \forall \nu \in \{1, \dots, n\} .$$

Then the assumptions made for the considered theory and the AB imply existence of real numbers φ_\pm with

$$\lim_{t \rightarrow \begin{smallmatrix} + \\ - \end{smallmatrix} \infty} |t|^N \left\| \Psi_{\text{out}}^{(\text{in})}(\check{\chi}_1, \dots, \check{\chi}_n) - \exp\left(i \varphi_{\begin{smallmatrix} + \\ - \end{smallmatrix}}\right) \hat{A}_{1,t} \cdots \hat{A}_{n,t} \Omega_0 \right\| = 0 \quad (2.101)$$

for all $N \in \mathbb{N}$ and HRK sequences $\{\hat{A}_{\nu,t}\}_{t \in \mathbb{R}}$ for the $\hat{a}_0^(\check{\chi}_\nu)\Omega_0$.*

Sketch of proof: Obviously, the (2.101) holds for $n = 1$ with $\varphi_\pm = 1$. Therefore, it is sufficient to prove the theorem for $n = n'$ assuming it to be valid for $n < n'$:

Without restriction of generality we may assume existence of some $\epsilon >=$ with

$$M_{\check{\chi}_1} + \dots + M_{\check{\chi}_{n'}} - U_\epsilon(M_{\check{\chi}_1}) \dots - U_\epsilon(M_{\check{\chi}_{n'}}) \subset U_m(0) . \quad (2.102)$$

Let us first consider HRK sequences of the type given by Lemma 2.3.3 and prove the lemma by induction w.r.t. n . Exploiting the AB and the definition of HRK sequences we easily see that

$$\lim_{t \rightarrow \begin{smallmatrix} + \\ - \end{smallmatrix} \infty} |t|^N \left(\left\| \hat{A}_{n'-1,t} \hat{A}_{n'-1,t}^* \Psi_{\text{out}}^{(\text{in})}(\check{\chi}_1, \dots, \check{\chi}_{n'}) - \Psi_{\text{out}}^{(\text{in})}(\check{\chi}_1, \dots, \check{\chi}_{n'}) \right\|^2 - \left\| \hat{A}_{n'-1,t} \hat{A}_{n'-1,t}^* \Psi_{\text{out}}^{(\text{in})}(\check{\chi}_1, \dots, \check{\chi}_{n'-1}) - \Psi_{\text{out}}^{(\text{in})}(\check{\chi}_1, \dots, \check{\chi}_{n'-1}) \right\|^2 \right) = 0$$

holds for all $N \in \mathbb{N}$. Since we already know that there are real φ'_+, φ'_- with

$$\lim_{t \rightarrow \begin{smallmatrix} + \\ - \end{smallmatrix} \infty} |t|^N \left\| \exp\left(i \varphi'_{\begin{smallmatrix} + \\ - \end{smallmatrix}}\right) \hat{A}_{1,t} \cdots \hat{A}_{n'-1,t} \Omega_0 - \Psi_{\text{out}}^{(\text{in})}(\check{\chi}_1, \dots, \check{\chi}_{n'-1}) \right\| = 0$$

for all $N \in \mathbb{N}$, we conclude:

$$\begin{aligned} & \lim_{t \rightarrow \begin{smallmatrix} + \\ - \end{smallmatrix} \infty} |t|^N \left\| \hat{A}_{n'-1,t} \hat{A}_{n'-1,t}^* \Psi_{\text{out}}^{(\text{in})}(\check{\chi}_1, \dots, \check{\chi}_{n'}) - \Psi_{\text{out}}^{(\text{in})}(\check{\chi}_1, \dots, \check{\chi}_{n'}) \right\| \\ &= \lim_{t \rightarrow \begin{smallmatrix} + \\ - \end{smallmatrix} \infty} |t|^N \left\| \hat{A}_{n'-1,t} \hat{A}_{n'-1,t}^* \Psi_{\text{out}}^{(\text{in})}(\check{\chi}_1, \dots, \check{\chi}_{n'-1}) - \Psi_{\text{out}}^{(\text{in})}(\check{\chi}_1, \dots, \check{\chi}_{n'-1}) \right\| \\ &= \lim_{t \rightarrow \begin{smallmatrix} + \\ - \end{smallmatrix} \infty} |t|^N \left\| \hat{A}_{n'-1,t} \hat{A}_{n'-1,t}^* \exp\left(i \varphi'_{\begin{smallmatrix} + \\ - \end{smallmatrix}}\right) \hat{A}_{1,t} \cdots \hat{A}_{n'-1,t} \Omega_0 - \Psi_{\text{out}}^{(\text{in})}(\check{\chi}_1, \dots, \check{\chi}_{n'-1}) \right\| \\ &= \lim_{t \rightarrow \begin{smallmatrix} + \\ - \end{smallmatrix} \infty} |t|^N \left\| \exp\left(i \varphi'_{\begin{smallmatrix} + \\ - \end{smallmatrix}}\right) \hat{A}_{1,t} \cdots \hat{A}_{n'-1,t} \hat{A}_{n'-1,t}^* \hat{A}_{n'-1,t} \Omega_0 - \Psi_{\text{out}}^{(\text{in})}(\check{\chi}_1, \dots, \check{\chi}_{n'-1}) \right\| \\ &= \lim_{t \rightarrow \begin{smallmatrix} + \\ - \end{smallmatrix} \infty} |t|^N \left\| \exp\left(i \varphi'_{\begin{smallmatrix} + \\ - \end{smallmatrix}}\right) \hat{A}_{1,t} \cdots \hat{A}_{n'-1,t} \Omega_0 - \Psi_{\text{out}}^{(\text{in})}(\check{\chi}_1, \dots, \check{\chi}_{n'-1}) \right\| \\ &= 0 \quad \forall N \in \mathbb{N} . \end{aligned}$$

Since, obviously,

$$\Psi_{\text{(in)}}^{\text{out}}(\check{\chi}_1, \dots, \check{\chi}_{n'}) = \Psi_{\text{(in)}}^{\text{out}}(\check{\chi}_{\pi 1}, \dots, \check{\chi}_{\pi n'}) \quad \forall \pi \in S'_n$$

this implies

$$\lim_{\substack{t \rightarrow +\infty \\ (-)}} |t|^N \left\| \hat{A}_{\nu, t} \hat{A}_{\nu, t}^* \Psi_{\text{(in)}}^{\text{out}}(\check{\chi}_1, \dots, \check{\chi}_{n'}) - \Psi_{\text{(in)}}^{\text{out}}(\check{\chi}_1, \dots, \check{\chi}_{n'}) \right\| = 0$$

for all $\nu \in \{1, \dots, n'\}$, $N \in \mathbb{N}$ and hence, by iteration,

$$\lim_{\substack{t \rightarrow +\infty \\ (-)}} |t|^N \left\| \hat{A}_{1, t} \hat{A}_{1, t}^* \cdots \hat{A}_{n', t} \hat{A}_{n', t}^* \Psi_{\text{(in)}}^{\text{out}}(\check{\chi}_1, \dots, \check{\chi}_{n'}) - \Psi_{\text{(in)}}^{\text{out}}(\check{\chi}_1, \dots, \check{\chi}_{n'}) \right\| = 0$$

for all $N \in \mathbb{N}$. Exploiting local commutativity, once again, this gives

$$\lim_{\substack{t \rightarrow +\infty \\ (-)}} |t|^N \left\| \hat{A}_{1, t} \cdots \hat{A}_{n', t} \hat{A}_{1, t}^* \cdots \hat{A}_{n', t}^* \Psi_{\text{(in)}}^{\text{out}}(\check{\chi}_1, \dots, \check{\chi}_{n'}) - \Psi_{\text{(in)}}^{\text{out}}(\check{\chi}_1, \dots, \check{\chi}_{n'}) \right\| = 0 \quad (2.103)$$

for all $N \in \mathbb{N}$. Now, for the HRK sequences of the type specified in Lemma 2.3.3 we have

$$\hat{A}_{1, t}^* \cdots \hat{A}_{n', t}^* \Psi_{\text{(in)}}^{\text{out}}(\check{\chi}_1, \dots, \check{\chi}_{n'}) \in \hat{E}(M_{\check{\chi}_1} + \dots + M_{\check{\chi}_{n'}} - U_\epsilon(M_{\check{\chi}_1}) \dots - U_\epsilon(M_{\check{\chi}_{n'}})) \mathcal{H},$$

since

$$\Psi_{\text{(in)}}^{\text{out}}(\check{\chi}_1, \dots, \check{\chi}_{n'}) \in \hat{E}(M_{\check{\chi}_1} + \dots + M_{\check{\chi}_{n'}}) \mathcal{H}.$$

By (2.102), the spectrum condition, and Borchers' theorem, therefore,

$$\hat{A}_{1, t}^* \cdots \hat{A}_{n', t}^* \Psi_{\text{(in)}}^{\text{out}}(\check{\chi}_1, \dots, \check{\chi}_{n'}) \sim \Omega \sim \Omega_0.$$

This together with (2.103) shows that there is a complex-valued function $\rho(t)$ with

$$\lim_{\substack{t \rightarrow +\infty \\ (-)}} |t|^N \left\| \rho(t) \hat{A}_{1, t} \cdots \hat{A}_{n', t} \Omega_0 - \Psi_{\text{(in)}}^{\text{out}}(\check{\chi}_1, \dots, \check{\chi}_{n'}) \right\| = 0 \quad \forall n \in \mathbb{N}.$$

Since, obviously,

$$\lim_{\substack{t \rightarrow +\infty \\ (-)}} \left\| \rho(t) \hat{A}_{1, t} \cdots \hat{A}_{n', t} \Omega_0 \right\| = 1 \quad \lim_{\substack{t \rightarrow +\infty \\ (-)}} \left\| \hat{A}_{1, t} \cdots \hat{A}_{n', t} \Omega_0 \right\|,$$

we are left to prove

$$\lim_{\substack{t \rightarrow +\infty \\ (-)}} |t|^N \sup_{s > t} \left\| \hat{A}_{1, t} \cdots \hat{A}_{n', t} \Omega_0 - \hat{A}_{1, s} \cdots \hat{A}_{n', s} \Omega_0 \right\| = 0 \quad \forall n \in \mathbb{N},$$

as far as the special HRK sequences are concerned. This however, is a simple consequence of

$$\lim_{t \rightarrow \begin{smallmatrix} + \\ - \end{smallmatrix} \infty} |t|^N \sup_{s, s' \in (t, t+1)} \left\| \hat{A}_{1,s} \cdots \hat{A}_{\nu, s'} \cdots \hat{A}_{n', s} \Omega_0 - \hat{A}_{1,s} \cdots \hat{A}_{n', s} \Omega_0 \right\| = 0 \quad \forall n \in \mathbb{N}.$$

Finally, it is an easy consequence of local commutativity and Definition 2.3.1 that

$$\lim_{t \rightarrow \begin{smallmatrix} + \\ - \end{smallmatrix} \infty} |t|^N \left\| \hat{A}_{1,t} \cdots \hat{A}_{n,t} \Omega_0 - \hat{A}_{1,t} \cdots \hat{A}'_{\nu,t} \cdots \hat{A}_{n,t} \Omega_0, t \right\|$$

holds for all $N \in \mathbb{N}$ and $\nu \in \{1, \dots, n\}$, if $\{\hat{A}'_{\nu,t}\}_{t \in \mathbb{R}}$ is any other HRK sequence for $\hat{a}_0^*(\check{\chi}_\nu) \Omega_0$; i.e.:

(2.101) does not depend on the special choice for the HRK sequences. ■

Exercise 41 Show the following:

- (i) Unitary mappings $\hat{V}_{\text{out}}^{\text{(in)}}$ from \mathcal{H}_0 onto \mathcal{H} fulfilling (2.86), (2.88), and the asymptotic condition do exist.
- (ii) The numbers φ_\pm may⁷¹ depend on n but not on the functions $\check{\chi}_\nu$.

Exercise 42 Assuming that the local algebras are given by a neutral scalar WIGHTMAN field as described above, show the following for the PCT operator $\hat{\theta}$ considered in Exercise 39 and the corresponding operator $\hat{\theta}_0$ of the FS:⁷²

- (i)

$$\hat{\theta} \hat{U}(a, \Lambda) = \hat{U}(-a, \Lambda) \hat{\theta} \quad \forall (a, \Lambda) \in \mathcal{P}_+^\uparrow,$$
- (ii)

$$\hat{\theta} \hat{V}_{\text{out}} = \hat{V}_{\text{in}} \hat{\theta}_0, \quad \hat{\theta} \hat{V}_{\text{in}} = \hat{V}_{\text{out}} \hat{\theta}_0,$$

- (iii) The described scattering theory is PCT-invariant in the sense that⁷³

$$\hat{\theta} \hat{S} = \hat{S}^{-1} \hat{\theta}.$$

⁷¹Suitable extension of the asymptotic condition implies $\varphi_\pm = 1$ for all n (see (Lücke, 1983)).

⁷²Note that (i) implies $\hat{\theta} \mathcal{A}(\mathcal{O}) = \mathcal{A}(-\mathcal{O}) \hat{\theta}$ for all open $\mathcal{O} \subset \mathbb{R}^4$.

⁷³For a (complicated) proof not using this assumption see (Epstein, 1967).

2.3.4 Cluster Properties of the S-Matrix

Definition 2.3.5 Let $\{M_t\}_{t \in \mathbb{R}} \subset \mathbb{R}^4$ and $\{\hat{B}_t\}_{t \in \mathbb{R}} \subset \mathcal{L}(\mathcal{H})$. Then \hat{B}_t is called **asymptotically localized** in M_t if the following two conditions are fulfilled:

(i) For every $N \in \mathbb{N}$ there is a sequence of local operators

$$\hat{A}_t \in \mathcal{A}\left(U_{\frac{|t|}{N}}(M_t)\right)$$

with

$$\lim_{t \rightarrow \pm\infty} |t|^N \left\| \hat{A}_t - \hat{B}_t \right\| = 0.$$

(ii)

$$\lim_{t \rightarrow \pm\infty} |t|^{-N} \left\| \hat{B}_t \right\| = 0 \quad \text{for sufficiently large } N \in \mathbb{N}.$$

An immediate consequence of Definition 2.3.5 is the following

Corollary 2.3.6 For $\nu = 1, 2$ let $\hat{B}_{j,t}$ be asymptotically localized in $M_{j,t}$. Then $\hat{B}_t \stackrel{\text{def}}{=} \hat{B}_{1,t} \hat{B}_{2,t}$ is asymptotically localized in $M_t \stackrel{\text{def}}{=} M_{1,t} \cup M_{2,t}$.

Definition 2.3.7 Let $\{M_t\}_{t \in \mathbb{R}} \subset \mathbb{R}^4$ and $\Psi \in \mathcal{H}$. Then $\{\hat{B}_t\}_{t \in \mathbb{R}} \subset \mathcal{L}(\mathcal{H})$ is called a M_t -**sequence** for Ψ if the following two conditions are fulfilled:⁷⁴

(i) $\lim_{t \rightarrow \pm\infty} |t|^N \left\| \hat{B}_t \Omega_0 - \Psi \right\| = 0 \quad \forall N \in \mathbb{N}$.

(ii) \hat{B}_t is asymptotically localized in M_t .

Theorem 2.3.8 Let $\check{\chi}_1, \dots, \check{\chi}_n \in \mathcal{D}(\mathbb{R}^3)$ be non-overlapping and let Σ be a smooth spacelike hypersurface (without finite boundary points) above resp. below⁷⁵ all the sets

$$(K_{\check{\chi}_\nu} + a_\nu) \cap (K_{\check{\chi}_\mu} + a_\mu), \quad \nu \neq \mu,$$

for given $a_1, \dots, a_n \in \mathbb{R}^4 \setminus \Sigma$. Then there are $(t\Sigma - ta_\nu)$ -sequences $\{\hat{B}_{\nu,t}\}_{t \in \mathbb{R}}$ for the 1-particle states $\hat{a}_0^*(\check{\chi}_\nu)\Omega_0$ and a real number φ_{out} resp. φ_{in} with⁷⁶

$$\lim_{\lambda \rightarrow +\infty} |t|^N \left\| \Psi_{\text{ex}} \left(\hat{U}_0(\lambda a_1) \check{\chi}_1, \dots, \hat{U}_0(\lambda a_n) \check{\chi}_n \right) - e^{i\varphi_{\text{ex}}} \hat{B}_{1,\lambda}(\lambda a_1) \cdots \hat{B}_{n,\lambda}(\lambda a_n) \Omega_0 \right\| = 0 \quad \forall N \in \mathbb{N}$$

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⁷⁴Note that for $\Psi = \hat{a}_0^*(\check{\chi})\Omega_0$ and $M_t = K_{\check{\chi}} \cap \Sigma_t$ the M_t -sequences are just HRK-sequences.

⁷⁵We say Σ is above a set $M \subset \mathbb{R}^4$ if $M \subset \left\{ (x^0 - t, \mathbf{x}) : x \in \Sigma, t > 0 \right\}$.

⁷⁶The special choice for the $(t\Sigma - ta_\nu)$ -sequences is not essential. As usual, we use the notation $\hat{B}(x) = \hat{U}(x) \hat{B} \hat{U}(x)^{-1}$ for $\hat{B} \in \mathcal{L}(\mathcal{H})$, $x \in \mathbb{R}^4$.

where $ex=out$ resp. $ex=in$.

Proof: Straightforward application of the described standard techniques. ■

Theorem 2.3.9 (Fredenhagen) *Let $\tau > 0$ and let $\hat{A}, \hat{B} \in \mathcal{L}(\mathcal{H})$ fulfill the condition⁷⁷*

$$[\hat{U}(t)\hat{A}\hat{U}(t)^{-1}, \hat{B}]_- = 0 \quad \forall t \in [-\tau, +\tau].$$

Then

$$\left| \langle \Omega_0 | \hat{A}\hat{B}\Omega_0 \rangle - \langle \Omega_0 | \hat{A}\Omega_0 \rangle \langle \Omega_0 | \hat{B}\Omega_0 \rangle \right| \leq e^{-m\tau} \sqrt{\|\hat{A}^*\Omega_0\| \|\hat{B}\Omega_0\| \|\hat{A}\Omega_0\| \|\hat{B}^*\Omega_0\|}.$$

Proof: See (Fredenhagen, 1985). ■

All we need for the derivation of cluster properties of the S-matrix is the following immediate consequence of Theorem 2.3.9.

Corollary 2.3.10 *For $\nu = 1, 2$ let $\hat{B}_{j,t}$ be asymptotically localized in $M_{j,t}$. If there is some ϵ for which*

$$\lambda > \frac{1}{\epsilon} \implies M_{1,\lambda} \times U_{\epsilon\lambda}(M_{2,\lambda}),$$

then

$$\lim_{\lambda \rightarrow +\infty} \lambda^N \left\| \langle \Omega_0 | \hat{B}_{1,\lambda} \hat{B}_{2,\lambda} \Omega_0 \rangle - \langle \Omega_0 | \hat{B}_{1,\lambda} \Omega_0 \rangle \langle \Omega_0 | \hat{B}_{2,\lambda} \Omega_0 \rangle \right\| = 0 \quad \forall N \in \mathbb{N}.$$

Now the cluster properties of the S-matrix are an immediate consequence:

Corollary 2.3.11 *Let $\check{\chi}_1, \dots, \check{\chi}_n, a_1, \dots, a_n$ and Σ be given as in Theorem 2.3.8 for ‘above’. Moreover, let $\check{\chi}'_1, \dots, \check{\chi}'_{n'}, a'_1, \dots, a'_{n'}$ and Σ' be given as in Theorem 2.3.8 for ‘below’. Finally, let $I \subset \{1, \dots, n\}$ and $I' \subset \{1, \dots, n'\}$ be such that*

$$\left((K_{\check{\chi}_{\nu_1}} + a_{\nu_1}) \cap \Sigma \right) \cup \left(\left(K_{\check{\chi}'_{\nu'_1}} + a'_{\nu'_1} \right) \cap \Sigma' \right)$$

is spacelike relative to

$$\left((K_{\check{\chi}_{\nu_2}} + a_{\nu_2}) \cap \Sigma \right) \cup \left(\left(K_{\check{\chi}'_{\nu'_2}} + a'_{\nu'_2} \right) \cap \Sigma' \right)$$

for all $\nu_1 \in I, \nu'_1 \in I'$ and $\nu_2 \in \{1, \dots, n\} \setminus I, \nu'_2 \in \{1, \dots, n'\} \setminus I'$.

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⁷⁷As usual, we write $\hat{U}(t)$ for $\hat{U}((t, 0, 0, 0), \mathbb{1}_4)$.

Then there is a real number ρ with

$$\lim_{\lambda \rightarrow +\infty} \lambda^N \left| e^{i\rho} \langle \Psi_{\text{out}}(\lambda) | \Psi_{\text{in}}(\lambda) \rangle - \langle \Psi_{1,\text{out}}(\lambda) | \Psi_{1,\text{in}}(\lambda) \rangle \langle \Psi_{2,\text{out}}(\lambda) | \Psi_{2,\text{in}}(\lambda) \rangle \right| = 0$$

for all $N \in \mathbb{N}$, where

$$\begin{aligned} \Psi_{\text{out}}(\lambda) &\stackrel{\text{def}}{=} \hat{V}_{\text{out}} \left(\prod_{\nu' \in \{1, \dots, n'\}} \left(\hat{U}(\lambda a'_{\nu'}) \hat{a}_0^*(\check{\chi}'_{\nu'}) \left(\hat{U}(\lambda a'_{\nu'}) \right)^{-1} \right) \Omega_0 \right), \\ \Psi_{1,\text{out}}(\lambda) &\stackrel{\text{def}}{=} \hat{V}_{\text{out}} \left(\prod_{\nu' \in I'} \left(\hat{U}(\lambda a'_{\nu'}) \hat{a}_0^*(\check{\chi}'_{\nu'}) \left(\hat{U}(\lambda a'_{\nu'}) \right)^{-1} \right) \Omega_0 \right), \\ \Psi_{2,\text{out}}(\lambda) &\stackrel{\text{def}}{=} \hat{V}_{\text{out}} \left(\prod_{\nu' \in \{1, \dots, n'\} \setminus I'} \left(\hat{U}(\lambda a'_{\nu'}) \hat{a}_0^*(\check{\chi}'_{\nu'}) \left(\hat{U}(\lambda a'_{\nu'}) \right)^{-1} \right) \Omega_0 \right), \\ \Psi_{\text{in}}(\lambda) &\stackrel{\text{def}}{=} \hat{V}_{\text{in}} \left(\prod_{\nu \in \{1, \dots, n\}} \left(\hat{U}(\lambda a_{\nu}) \hat{a}_0^*(\check{\chi}_{\nu}) \left(\hat{U}(\lambda a_{\nu}) \right)^{-1} \right) \Omega_0 \right), \\ \Psi_{1,\text{in}}(\lambda) &\stackrel{\text{def}}{=} \hat{V}_{\text{in}} \left(\prod_{\nu \in I} \left(\hat{U}(\lambda a_{\nu}) \hat{a}_0^*(\check{\chi}_{\nu}) \left(\hat{U}(\lambda a_{\nu}) \right)^{-1} \right) \Omega_0 \right), \\ \Psi_{2,\text{in}}(\lambda) &\stackrel{\text{def}}{=} \hat{V}_{\text{in}} \left(\prod_{\nu \in \{1, \dots, n\} \setminus I} \left(\hat{U}(\lambda a_{\nu}) \hat{a}_0^*(\check{\chi}_{\nu}) \left(\hat{U}(\lambda a_{\nu}) \right)^{-1} \right) \Omega_0 \right). \end{aligned}$$

Typical consequences of Corollary 2.3.11 are illustrated by Figures 2.9 and 2.8 (see also (Lücke, 8384)).⁷⁸ Roughly speaking, Corollary 2.3.11 shows that, from the macroscopic point of view, the S-matrix does not violate the causality principle.

It should have become clear from the evaluation of the asymptotic condition, that the field $\hat{\Phi}(x)$ itself is not necessary to determine the S-matrix once the set of almost local operators is determined. Note that the representation $\hat{U}(a, \Lambda)$ of \mathcal{P}_+^\uparrow is already fixed, up to unitary equivalence, by the FS. Of course, one cannot expect that the Hamiltonian \hat{P}^0 itself determines the physical picture of the dynamics unless the physical interpretation of the other observables is sufficiently well established. The above considerations show that it is sufficient to specify the macroscopically localized observables consistently, in order to select the S-matrix.

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⁷⁸Figure 2.8 illustrates the case $n = 4$, $n' = 6$, $I = I' = \{1, 2\}$; using the notation $K_\nu \stackrel{\text{def}}{=} K_{\check{\chi}_\nu} + \lambda a_\nu$, $K'_\nu \stackrel{\text{def}}{=} K_{\check{\chi}'_\nu} + \lambda a'_\nu$

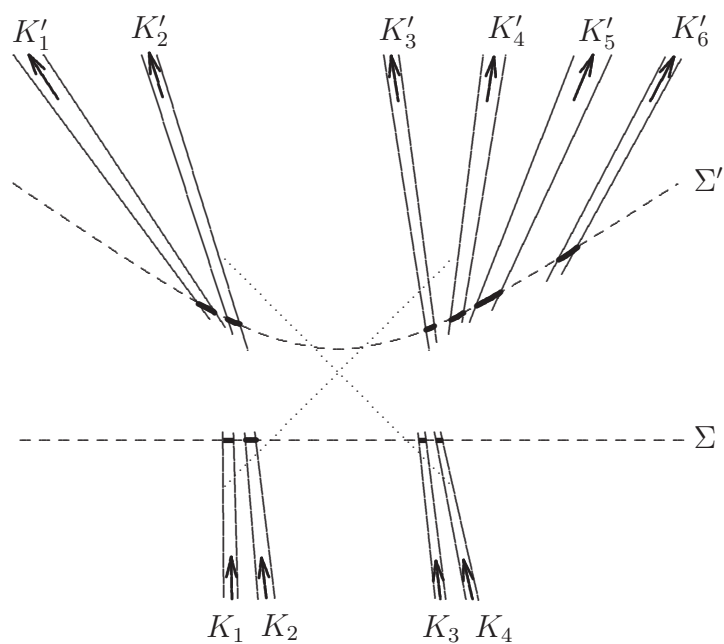


Figure 2.8: Macroscopically independent: ‘ $1+2 \longrightarrow 1+2$ ’ and ‘ $3+4 \longrightarrow 3+\dots+6$ ’

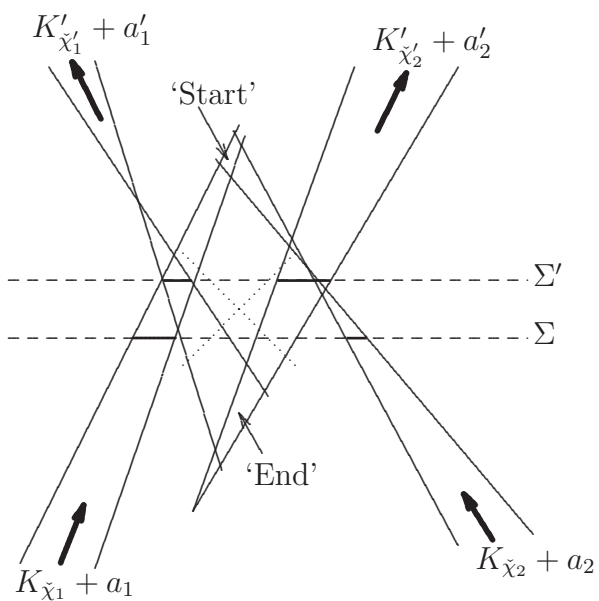


Figure 2.9: Macroscopically forbidden: ‘output before input’

2.4 Charged Scalar Fields

2.4.1 Free Charged Scalar Fields

Fields Operators

It is known, **nowadays**, that for every charged particle there is an *antiparticle* with opposite charge. Let (2.39) describe such a particle in the sense of 2.1.3 and, similarly,

$$\check{\Phi}_0^{(+)}(x) \stackrel{\text{def}}{=} (2\pi)^{-3/2} \int_{p^0=\omega_{\mathbf{p}}} \check{a}_0(\mathbf{p}) e^{-ipx} \frac{d\mathbf{p}}{2p^0}$$

the corresponding antiparticle on its FOCK space $\check{\mathcal{H}}_0$ with domain \check{D}_0 and representation $\check{U}_0(a, \Lambda)$ of \mathcal{P}_+^\dagger . Then both particles may be described simultaneously by the **charged scalar field**

$$\hat{\Phi}_q(x) \stackrel{\text{def}}{=} \hat{\Phi}_0^{(+)}(x) \otimes \mathbb{1} + \left(\mathbb{1} \otimes \check{\Phi}_0^{(+)}(x) \right)^* \quad (2.104)$$

on

$$D_q \stackrel{\text{def}}{=} D_0 \otimes \check{D}_0 \subset \mathcal{H}_q \stackrel{\text{def}}{=} \mathcal{H}_0 \otimes \check{\mathcal{H}}_0,$$

generalizing (2.43). As for the neutral scalar field, the relations⁷⁹

$$(\square + m^2) \hat{\Phi}_q(x) = 0, \quad (2.105)$$

$$\hat{U}_q(a, \Lambda) \hat{\Phi}_q(x) \hat{U}_q(a, \Lambda)^{-1} = \hat{\Phi}_q(\Lambda x + a), \quad (2.106)$$

$$\left[\hat{\Phi}_q(x), \hat{\Phi}_q(y) \right]_- = 0 \text{ for } x \times y, \quad (2.107)$$

are fulfilled for the charged field, where

$$\hat{U}_q(a, \Lambda) \stackrel{\text{def}}{=} \hat{U}_0(a, \Lambda) \otimes \check{U}_0(a, \Lambda).$$

Nevertheless, **the charged field cannot be interpreted as observable of whatever field strength** since, according to (2.104), it is not hermitian. This is the price to be paid for the important commutation relation

$$\left[\hat{Q}, \hat{\Phi}_q(x) \right]_- = -q \hat{\Phi}_q(x), \quad \text{where:} \quad (2.108)$$

\hat{Q} = observable of the **additive** charge,

q = charge of the particle,

$-q$ = charge of the antiparticle,

valid on a suitable domain. The main purpose of the **charged** scalar field is to create a dense set of well-interpreted states (for scattering theory) out of the vacuum.

Now it is important to supplement (2.107) by⁸⁰

$$\left[\hat{\Phi}_q(x), \hat{\Phi}_q^*(y) \right]_- = 0 \text{ for } x \times y. \quad (2.109)$$

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⁷⁹(2.107) holds for all x, y .

⁸⁰More precisely, (2.46) becomes $\left[\hat{\Phi}_q(x), \hat{\Phi}_q^*(y) \right]_- = i\Delta(x - y)$.

Local Gauge Transformations

Since the charged scalar field $\hat{\Phi}_q(x)$ is non-hermitian, anyway, it may well be replaced by

$$\hat{\Phi}_q^\lambda(x) \stackrel{\text{def}}{=} e^{i\hat{Q}\lambda(x)} \hat{\Phi}_q(x) e^{-i\hat{Q}\lambda(x)} = e^{-iq\lambda(x)} \hat{\Phi}_q(x) \quad (2.110)$$

(2.108)

if, simultaneously, the KLEIN-GORDON equation (2.105) is replaced by

$$((\partial_\mu + iq(\partial_\mu\lambda)(x)) (\partial^\mu + iq(\partial^\mu\lambda)(x)) + m^2) \hat{\Phi}_q^\lambda(x) = 0, \quad (2.111)$$

where $\lambda(x)$ denotes an arbitrary, but sufficiently smooth, real-valued function on \mathbb{R}^4 . As a direct generalization this leads to the KLEIN-GORDON equation

$$((\partial_\mu + iqA_\mu(x)) (\partial^\mu + iqA^\mu(x)) + m^2) \hat{\Phi}^\lambda(x) = 0 \quad (2.112)$$

for the quantized scalar field $\hat{\Phi}(x)$ interacting with the *external* (classical) electromagnetic field $A^\mu(x)$, being invariant under ***gauge transformations of second kind***

$$\hat{\Phi}(x) \longrightarrow e^{-iq\lambda(x)} \hat{\Phi}, \quad A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu\lambda(x). \quad (2.113)$$

Consistency Considerations

Exchanging the roles of particles and antiparticles results, according to (2.104), in the transition

$$\hat{\Phi}_{(q)}(x) \longrightarrow \hat{\Phi}_{(q)}^*(x), \quad q \longrightarrow -q.$$

Obviously, (2.112) is invariant under this transformation (since $A^\mu(x)$ is real).

For sufficiently well-behaved external fields $A^\mu(x)$ there exist solutions $\hat{\Phi}$ of (2.112) respecting the interaction picture (Seiler, 1978).

Interpreting classical solutions of the KLEIN-GORDON equation as expectation values of the quantized KLEIN-GORDON field solves the well-known problem raised by creation of negative frequency contributions in certain scattering problems (see e.g. (Baym, 1969, Chapt. 22)).

Replacing the classical electromagnetic potential $A^\mu(x)$ in (2.112) by the corresponding quantum field $\hat{A}^\mu(x)$ leads to the difficult problem of defining products of quantized fields (basic problems of *renormalization theory*).

2.4.2 WIGHTMAN Theory for Charged Scalar Fields

WIGHTMAN Axioms

A WIGHTMAN *theory* of a single **charged** scalar field $\hat{\Phi}(x)$ is characterized by the following assumptions (WIGHTMAN *axioms*):

0. Assumptions of Relativistic Quantum Theory:

Exactly the same as those for the WIGHTMAN Theory of a single **neutral** scalar field, as formulated in Section 2.2.1.

I. Assumptions about the Domain and Continuity of the Field:

There are two fields $\hat{\Phi}(x)$ and $\hat{\Phi}^*(x)$ defined as operator-valued, tempered, generalized functions with invariant domain $D \subset \mathcal{H}$; i.e. **linear** mappings

$$\begin{aligned} \hat{\Phi} : \mathcal{S}(\mathbb{R}^4) &\longrightarrow L(D, D) \\ \varphi &\longmapsto \hat{\Phi}(\varphi) = \underbrace{\int \hat{\Phi}(x)\varphi(x) dx}_{\text{formal}} \end{aligned}$$

and

$$\begin{aligned} \hat{\Phi}^* : \mathcal{S}(\mathbb{R}^4) &\longrightarrow L(D, D) \\ \varphi &\longmapsto \hat{\Phi}^*(\varphi) = \underbrace{\int \hat{\Phi}^*(x)\varphi(x) dx}_{\text{formal}} \end{aligned}$$

for which all the

$$\int \langle \Psi | \hat{\Phi}(x)\Psi \rangle \varphi(x) dx \stackrel{\text{def}}{=} \langle \Psi | \hat{\Phi}(\varphi)\Psi \rangle, \Psi \in D,$$

and

$$\int \langle \Psi | \hat{\Phi}^*(x)\Psi \rangle \varphi(x) dx \stackrel{\text{def}}{=} \langle \Psi | \hat{\Phi}^*(\varphi)\Psi \rangle, \Psi \in D,$$

are **continuous** in $\varphi \in \mathcal{S}(\mathbb{R}^4)$, where D has to fulfill the following conditions for $\varphi \in \mathcal{S}(\mathbb{R}^4)$ and $(a, \Lambda) \in \mathcal{P}_+^\uparrow$:

$$\Omega \subset D, \quad \hat{U}(a, \Lambda)D \subset D, \quad \hat{\Phi}(\varphi)D \subset D \supset \hat{\Phi}^*(\varphi)D.$$

The fields $\hat{\Phi}(x)$ and $\hat{\Phi}^*(x)$ are related by

$$\langle \Psi | \hat{\Phi}^*(\varphi)\Psi \rangle = \langle \Psi | \left(\hat{\Phi}(\bar{\varphi}) \right)^* \Psi \rangle \quad \forall \Psi \in D, \varphi \in \mathcal{S}(\mathbb{R}^4). \quad (2.114)$$

II. Transformation Law of the Field:

The field operators $\hat{\Phi}(x)$ and $\hat{\Phi}^*(x)$ transform according to

$$\hat{U}(a, \Lambda)\hat{\Phi}(x)\hat{U}(a, \Lambda)^{-1} = \hat{\Phi}(\Lambda x + a) \quad \forall (a, \Lambda) \in \mathcal{P}_+^\uparrow$$

and⁸¹

$$\hat{U}(a, \Lambda) \hat{\Phi}^*(x) \hat{U}(a, \Lambda)^{-1} = \hat{\Phi}^*(\Lambda x + a) \quad \forall (a, \Lambda) \in \mathcal{P}_+^\uparrow.$$

III. Local Commutativity:

The smeared fields $\hat{\Phi}(\varphi_1)$ and $\hat{\Phi}(\varphi_2)$ resp. $\hat{\Phi}^*(\varphi_2)$ commute whenever the supports of the test functions $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^4)$ are spacelike with respect to each other.⁸² Formally:

$$x \times y \implies [\hat{\Phi}(x), \hat{\Phi}(y)]_- = [\hat{\Phi}(x), \hat{\Phi}^*(y)]_- = 0.$$

Finally, the vacuum vector Ω is required to be **cyclic** with respect to the algebra \mathcal{F}_0 generated by $\hat{1} \wedge D$ and the smeared field operators $\hat{\Phi}(\varphi)$ and $\hat{\Phi}^*(\varphi)$ with $\varphi \in \mathcal{S}(\mathbb{R}^4)$:

$$D_0 \stackrel{\text{def}}{=} \mathcal{F}_0 \Omega \text{ is dense in } \mathcal{H}.$$

Obviously, all these axioms are fulfilled for the free **charged** field $\hat{\Phi}(x) = \hat{\Phi}_q(x)$, if we define $D \stackrel{\text{def}}{=} D_q$ and

$$\hat{\Phi}^*(\varphi) \stackrel{\text{def}}{=} \left(\hat{\Phi}(\bar{\varphi}) \right)^* \wedge D \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^4).$$

PCT and Spin-Statistics Theorem

The ‘connection between spin and statistics’ for the theory of a single charged field is given by the following two theorems.

Theorem 2.4.1 *There is no charged field $\hat{\Phi}(x) \neq 0$, fulfilling all the WIGHTMAN axioms with the possible exception of local commutativity, for which*

$$x \times y \implies \hat{\Phi}(x) \hat{\Phi}^*(y) + \hat{\Phi}^*(y) \hat{\Phi}(x) = 0.$$

Sketch of proof: The techniques used in Section 2.2.4 show that the expectation values of products of field operators exist as generalized functions and that there are L_+^\uparrow -invariant generalized functions $W, \check{W} \in \mathcal{S}(\mathbb{R}^4)'$ with

$$\begin{aligned} \left\langle \Omega \mid \hat{\Phi}(x) \hat{\Phi}^*(y) \Omega \right\rangle &= W(x - y), \\ \left\langle \Omega \mid \hat{\Phi}^*(x) \hat{\Phi}(y) \Omega \right\rangle &= \check{W}(x - y), \end{aligned}$$

⁸¹By (2.114) the transformation law of $\hat{\Phi}(x)$ implies that of $\hat{\Phi}^*(x)$ and vice versa.

⁸²Note that that this condition can be shown to be necessary to avoid acausal effects.

and

$$\text{supp}\widetilde{W} \subset \overline{V_+} \supset \text{supp}\widetilde{W}. \quad (2.115)$$

Now, assume

$$x \times y \implies \hat{\Phi}(x)\hat{\Phi}^*(y) + \hat{\Phi}^*(y)\hat{\Phi}(x) = 0,$$

i.e.

$$W(\xi) + \check{W}(-\xi) = 0 \quad \text{for } \xi \times 0.$$

Since $\check{W}(\xi) - \check{W}(-\xi)$, being an odd L_+^\uparrow -invariant distribution, vanishes for spacelike ξ this implies

$$F(\xi) \stackrel{\text{def}}{=} W(\xi) + \check{W}(\xi) = 0 \quad \text{for } \xi \times 0.$$

Since, by (2.115),

$$\text{supp}\check{F} \subset \overline{V_+} \supset \text{supp}\widetilde{W}$$

Corollary 2.2.9 tells us that $F = 0$, i.e:

$$\langle \Omega | \hat{\Phi}(x)\hat{\Phi}^*(y)\Omega \rangle + \langle \Omega | \hat{\Phi}^*(-y)\hat{\Phi}(-x)\Omega \rangle = 0.$$

Therefore, we have

$$\begin{aligned} 0 &= \int \left(\langle \Omega | \hat{\Phi}(x)\hat{\Phi}^*(y)\Omega \rangle + \langle \Omega | \hat{\Phi}^*(-y)\hat{\Phi}(-x)\Omega \rangle \right) \varphi(x)\overline{\varphi(y)} \, dx dy \\ &= \left\| \left(\hat{\Phi}(\varphi) \right)^* \Omega \right\|^2 + \left\| \int \hat{\Phi}(x)\varphi(-x) \, dx \Omega \right\|^2 \end{aligned}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^4)$ and hence

$$\hat{\Phi}^*(x)\Omega = \hat{\Phi}(x)\Omega = 0.$$

By cyclicity of Ω , however, this would imply $\hat{\Phi}(x) = 0$. ■

Theorem 2.4.2 *There is no charged field $\hat{\Phi}(x) \neq 0$, fulfilling all the WIGHTMAN axioms with the possible exception of local commutativity, for which the conditions*

$$x \times y \implies \hat{\Phi}(x)\hat{\Phi}^*(y) - \hat{\Phi}^*(y)\hat{\Phi}(x) = 0$$

and

$$x \times y \implies \hat{\Phi}(x)\hat{\Phi}(y) + \hat{\Phi}(y)\hat{\Phi}(x) = 0$$

hold.

Sketch of Proof:⁸³ Assume

$$x \times y \implies \hat{\Phi}(x)\hat{\Phi}^*(y) - \hat{\Phi}^*(y)\hat{\Phi}(x) = 0$$

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⁸³This proof may be applied to a much more general situation (see (Streater and Wightman, 1989, Theorem 4.8)).

and

$$x \times y \implies \hat{\Phi}(x)\hat{\Phi}(y) + \hat{\Phi}(y)\hat{\Phi}(x) = 0.$$

Then for all $\varphi, \psi \in \mathcal{D}(\mathbb{R}^4)$ with

$$\text{supp}\varphi \times \text{supp}\psi$$

we have

$$\begin{aligned} 0 &\leq \left\| \hat{\Phi}^*(\varphi)\hat{\Phi}(\psi)\Omega \right\|^2 \\ &= \left\langle \Omega \mid \left(\hat{\Phi}(\psi) \right)^* \left(\hat{\Phi}^*(\varphi) \right)^* \hat{\Phi}^*(\varphi)\hat{\Phi}(\psi)\Omega \right\rangle \\ &= - \left\langle \Omega \mid \left(\hat{\Phi}^*(\varphi) \right)^* \hat{\Phi}^*(\varphi) \left(\hat{\Phi}(\psi) \right)^* \hat{\Phi}(\psi)\Omega \right\rangle. \end{aligned}$$

Since

$$\hat{U}(a)\hat{\Phi}(\psi)\hat{U}(a)^{-1} = \int \hat{\Phi}(x)\psi(x-a) dx \quad \forall a \in \mathbb{R}^4,$$

this implies

$$\left\langle \Omega \mid \left(\hat{\Phi}^*(\varphi) \right)^* \hat{\Phi}^*(\varphi)\hat{U}(\lambda a) \left(\hat{\Phi}(\psi) \right)^* \hat{\Phi}(\psi)\Omega \right\rangle \leq 0$$

for spacelike a and sufficiently large $\lambda = \lambda(a, \varphi, \psi)$. On the other hand, however, one may prove⁸⁴ that

$$\begin{aligned} &\lim_{\lambda \rightarrow \infty} \left\langle \Omega \mid \left(\hat{\Phi}^*(\varphi) \right)^* \hat{\Phi}^*(\varphi)\hat{U}(\lambda a) \left(\hat{\Phi}(\psi) \right)^* \hat{\Phi}(\psi)\Omega \right\rangle \\ &= \left\langle \Omega \mid \left(\hat{\Phi}^*(\varphi) \right)^* \hat{\Phi}^*(\varphi)\Omega \right\rangle \left\langle \Omega \mid \left(\hat{\Phi}(\psi) \right)^* \hat{\Phi}(\psi)\Omega \right\rangle \\ &= \left\| \hat{\Phi}^*(\varphi)\Omega \right\|^2 \left\| \hat{\Phi}(\psi)\Omega \right\|^2. \end{aligned}$$

Therefore

$$\hat{\Phi}^*(x)\Omega = \hat{\Phi}(x)\Omega = 0,$$

which, by cyclicity of Ω , would imply $\hat{\Phi}(x) = 0$. ■

Now, of course, the PCT theorem has to involve both $\hat{\Phi}(x)$ and $\hat{\Phi}^*(x)$:

Theorem 2.4.3 *Let $n \in \mathbb{N}$ and let $\hat{\Phi}(x)$ be a charged scalar field fulfilling all the WIGHTMAN axioms with the possible exception of local commutativity. Then the PCT condition*

$$\begin{aligned} \left\langle \Omega \mid \hat{\Phi}_1(x_1) \cdots \hat{\Phi}_n(x_n)\Omega \right\rangle &= \left\langle \Omega \mid \hat{\Phi}_n(-x_n) \cdots \hat{\Phi}_1(-x_1)\Omega \right\rangle \\ &\quad \forall x \in \mathbb{R}^{4n}, \hat{\Phi}_\nu \in \left\{ \hat{\Phi}, \hat{\Phi}^* \right\} \end{aligned} \tag{2.116}$$

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⁸⁴See (Araki et al., 1962, Theorem 3) for a proof of this *cluster property* not depending on 0 being an isolated point of the energy-momentum spectrum. See also (Maison, 1968) and, for a C^* algebraic version, (Baumgärtel, 1995, Theorem 1.2.5).

is equivalent to the condition of **weak local commutativity**

$$\begin{aligned} \left\langle \Omega \mid \hat{\Phi}_1(x_1) \cdots \hat{\Phi}_n(x_n) \Omega \right\rangle &= \left\langle \Omega \mid \hat{\Phi}_n(x_n) \cdots \hat{\Phi}_1(x_1) \Omega \right\rangle \\ \text{for } (x_1 - x_2, \dots, x_{n-1} - x_n) &\in \mathcal{J}_{n-1}, \hat{\Phi}_\nu \in \left\{ \hat{\Phi}, \hat{\Phi}^* \right\}. \end{aligned} \quad (2.117)$$

Proof: Analogous to that of Corollary 2.2.22

Exercise 43 Show that (2.116) is equivalent to existence of an anti-unitary Operator $\hat{\theta}$ fulfilling the conditions

$$\begin{aligned} \hat{\theta} \hat{\theta} &= \hat{1}, \quad \hat{\theta} \Omega = \Omega, \\ \hat{\theta} \hat{\Phi}(\varphi) \hat{\theta} &= \left(\int \hat{\Phi}(-x) \varphi(x) dx \right)^* \wedge D \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^4), \hat{\Phi} \in \left\{ \hat{\Phi}, \hat{\Phi}^* \right\} \end{aligned}$$

and

$$\hat{\theta} \hat{U}(a, \Lambda) \hat{\theta} = \hat{U}(-a, \Lambda) \quad \forall (a, \Lambda) \in \mathcal{P}_+^\uparrow.$$

2.4.3 Scattering Theory

Now assume that the IS is described by a charged scalar field $\hat{\Phi}(x)$ fulfilling all the WIGHTMAN axioms and that the corresponding FS may be described by the free charged scalar field $\hat{\Phi}_q(x)$.

Since the 1-particle states are charged they can no longer be approximated by states of the form $\hat{B}\Omega$, $\hat{B} \in \mathcal{A}_{\text{loc}}$. Here the local observable algebras have to be replaced by some net of **local field algebras** $\mathcal{F}_b(\mathcal{O})$. It is not evident how to define these algebras of **bounded** operators.⁸⁵ This problem can be avoided by working with unbounded operators (Lücke, 1983). For simplicity, however, let us assume that the $\mathcal{F}_b(\mathcal{O})$ are specified and fulfill the conditions of **isotony**

$$\mathcal{O}_1 \subset \mathcal{O}_2 \implies \mathcal{F}_b(\mathcal{O}_1) \subset \mathcal{F}_b(\mathcal{O}_2),$$

local commutativity

$$\mathcal{O}_1 \times \mathcal{O}_2 \implies [\mathcal{F}_b(\mathcal{O}_1), \mathcal{F}_b(\mathcal{O}_2)]_- = \{\hat{0}\},$$

POINCARÉ **covariance**

$$\hat{U}(a, \Lambda) \mathcal{F}_b(\mathcal{O}) \hat{U}(a, \Lambda)^{-1} = \mathcal{F}_b(\Lambda \mathcal{O} + a),$$

and **irreducibility**

$$\left(\bigcup_{R>0} \mathcal{F}_b(U_R(o)) \right)' = \{\lambda \hat{1} : \lambda \in \mathbb{C}\}$$

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⁸⁵See (Driessler et al., 1986a) for a detailed discussion of this point.

(compare (2.90)–(2.93)). Then the corresponding net of local observable algebras is given by

$$\mathcal{A}(\mathcal{O}) \stackrel{\text{def}}{=} \left\{ \hat{A} \in \mathcal{F}_b(\mathcal{O}) : [\hat{A}, \hat{Q}]_- = 0 \right\},$$

where \hat{Q} is the *charge operator* (uniquely) defined by

$$\left. \begin{aligned} [\hat{Q}, \hat{\Phi}(\varphi)]_- &= -q\hat{\Phi}(\varphi) \\ [\hat{Q}, \hat{\Phi}^*(\varphi)]_- &= +q\hat{\Phi}^*(\varphi) \end{aligned} \right\} \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^4), \quad \hat{Q}\Omega = 0, \quad \hat{Q}^* = \hat{Q},$$

and the scattering theory described for the neutral field can be applied to the charged field with the following modifications:

1. Everywhere, except in the asymptotic condition, the field algebras $\mathcal{F}_b(\mathcal{O})$ have to be used instead of the observable algebras.
2. Asymptotic states with arbitrary number of particles **and** antiparticles have to be considered.
3. The HRK sequences have to be defined for both particle and antiparticle states.

Now, of course, the resulting PCT invariance of the transition probabilities also involves interchange of particles and antiparticles.

Chapter 3

$\lambda\Phi_4^4$ Perturbation Theory

“Renormalization theory has a history of egregious errors by distinguished savants. It has a justified reputation for perversity; a method that works up to 13th order in the perturbation series fails in the 14th order. Arguments that sound plausible often dissolve into mush when examined closely. The worst that can happen often happens. The prudent student would do well to distinguish sharply between what has been proved and what has been made plausible, and in general he should watch out!”

A. S. WIGHTMAN ([Velo and Wightman, 1976](#), p. 16)

3.1 General Aspects

3.1.1 Interaction Picture

General Definition

Let us **formally** assume that the ‘same’ **instant** measurements can be performed on the IS as well as on the FS and that the expectation values for all such measurements performable at a fixed time t determine the corresponding state uniquely. Then we say

“ $\check{\Psi} \in \check{\mathcal{H}}$ resp. $\check{\Psi}_0 \in \check{\mathcal{H}}_0$ looks like $\check{\Phi}_0 \in \check{\mathcal{H}}_0$ at time t ”

if all the expectation values for identical measurements to be performed at time t predicted by $\check{\Psi}$ resp. $\check{\Psi}_0$ are the same as those predicted by $\check{\Phi}_0$.

Moreover, let us assume that for every state $\check{\Psi}$ and for every instant of time t there is a state $\check{\Psi}_I(t)$ of the FS such that $\check{\Psi} \in \check{\mathcal{H}}$ looks like $\check{\Psi}_I(t) \in \check{\mathcal{H}}_0$ at time t . Then we call $\check{\Psi}_I(t)$ the **instantaneous state** at time t of the IS **in the interaction picture**, if the IS is in the actual state $\check{\Psi}$. See Fig. 3.1 for the example of a classical particle moving in an external potential.

In general, if the interaction picture exists, the AC should be of the form

$$\check{\Psi}_I(t) \xrightarrow[t \rightarrow \pm\infty]{} \check{\Psi}_{\pm} \quad (3.1)$$

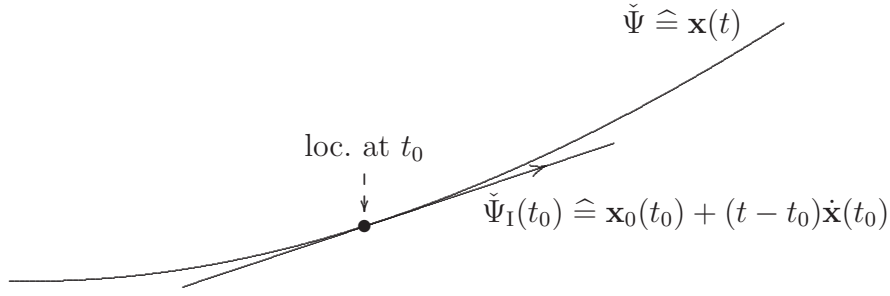


Figure 3.1: Interaction picture for a classical particle

for suitable specification of the type of convergence.

Formalization (in view of quantum theory)

Assuming that the interaction picture exists, in the sense described above, let us introduce the following notation:

$$\begin{aligned}
 \check{\mathcal{A}}_0 &\stackrel{\text{def}}{=} \text{set of all instantaneous measurement performable at time } 0, \\
 \check{\alpha}_{\Delta t}(A) &\stackrel{\text{def}}{=} \text{measurement } A \in \check{\mathcal{A}}_0 \text{ time-shifted by } \Delta t, \\
 \check{\mathcal{A}} &\stackrel{\text{def}}{=} \bigcup_{t \in \mathbb{R}} \{ \check{\alpha}_t(A) : A \in \check{\mathcal{A}}_0 \}, \\
 E(A, \check{\Psi}) &\stackrel{\text{def}}{=} \text{expectation value of } A \in \check{\mathcal{A}} \text{ for the IS in state } \check{\Psi}, \\
 E_0(A, \check{\Psi}_0) &\stackrel{\text{def}}{=} \text{expectation value of } A \in \check{\mathcal{A}} \text{ for the FS in state } \check{\Psi}_0.
 \end{aligned}$$

While the set of all instantaneous measurement procedures is the same for the IS and the FS, the instantaneous states develop differently in time:¹

$$\begin{aligned}
 \hat{U}(t)\check{\Psi} &\stackrel{\text{def}}{=} \text{state of the IS, at time } t \text{ looking like } \check{\Psi} \text{ at time } 0, \\
 \hat{U}_0(t)\check{\Psi}_0 &\stackrel{\text{def}}{=} \text{state of the FS, at time } t \text{ looking like } \check{\Psi}_0 \text{ at time } 0.
 \end{aligned} \tag{3.2}$$

Consistency requires

$$E_{(0)}(\check{\alpha}_t(A), \hat{U}_{(0)}(t)\check{\Psi}_{(0)}) = E_{(0)}(A, \check{\Psi}_{(0)}) \quad \forall A \in \check{\mathcal{A}}_0. \tag{3.3}$$

The instantaneous state $\check{\Psi}_I(t)$ in the interaction picture at time t of the IS in the actual state $\check{\Psi}$ is determined by

$$E_0(\check{\alpha}_t(A), \check{\Psi}_I(t)) = E(\check{\alpha}_t(A), \check{\Psi}) \quad \forall A \in \check{\mathcal{A}}_0. \tag{3.4}$$

Defining

$$\hat{W}(t)\check{\Psi} \stackrel{\text{def}}{=} \check{\Psi}_I(t),$$

¹**Warning:** In general, $\hat{U}(t)$ and $\hat{U}_0(t)$ depend on the choice for the origin of the time-scale.

by (3.1) we formally get

$$\hat{W}(t)\check{\Psi} \xrightarrow[t \rightarrow \pm\infty]{(-)} \check{\Psi}_{(-)} \stackrel{(2.78)}{=} \hat{V}_{(\text{in})}^{-1}\check{\Psi} \quad \forall \check{\Psi} \in \check{\mathcal{H}}_{(\text{in})}^{\text{out}},$$

i.e.:

$$\hat{W}(t)^{-1} \xrightarrow[t \rightarrow \pm\infty]{(-)} \hat{V}_{(\text{in})}^{\text{out}}. \quad (3.5)$$

By (3.3) and (3.4), assuming (without restriction of generality)

$$\hat{W}(0) = \text{identity mapping},$$

we have

$$\hat{W}(t) = \hat{U}_0(t)\hat{U}(t)^{-1}, \quad \hat{W}(t)^{-1} = \hat{U}(t)\hat{U}_0(t)^{-1} \quad (3.6)$$

and therefore, by (2.82), (3.5) implies

$$\boxed{\begin{aligned} \hat{S}_0 &= \lim_{t_{\pm} \rightarrow \pm\infty} \hat{\Omega}(t_+, t_-), \text{ where:} \\ \hat{\Omega}(t_+, t_-) &\stackrel{\text{def}}{=} \hat{U}_0(t_+)\hat{U}(t_+)^{-1}\hat{U}(t_-)\hat{U}_0(t_-)^{-1}. \end{aligned}} \quad (3.7)$$

The type of limit, depending on the model, has to be suitably defined, of course. If, in addition, we also have homogeneity in time, i.e.²

$$\hat{U}_{(0)}(t_1)\hat{U}_{(0)}(t_2) = \hat{U}_{(0)}(t_1 + t_2),$$

(3.5) and (3.6) imply

$$\hat{U}(t)\hat{V}_{(\text{in})}^{\text{out}} = \hat{V}_{(\text{in})}^{\text{out}}\hat{U}_0(t), \quad \hat{V}_{(\text{in})}^{\text{out}}\hat{U}(t) = \hat{U}_0(t)\hat{V}_{(\text{in})}^{\text{out}}. \quad (3.8)$$

and, by (2.79)/(2.82), therefore:

$$[\hat{S}_{(0)}, \hat{U}_{(0)}(t)]_- = 0.$$

Application to Quantum Theory

In quantum theory the states

$$\check{\Psi}_{(0)} = \omega_{\Psi_{(0)}}$$

are given by state vectors $\Psi_{(0)}$ from the corresponding Hilbert space $\mathcal{H}_{(0)}$ and the time translations $\hat{U}_{(0)}(t)$ are given by unitary operators $\hat{U}_{(0)}(t)$ in $\mathcal{H}_{(0)}$,

$$\hat{U}_{(0)}(t)\check{\Psi}_{(0)} = \omega_{\hat{U}_{(0)}(t)\Psi_{(0)}},$$

²In this case $\hat{U}(t)$ and $\hat{U}_0(t)$ are independent of the choice for the origin of the time-scale.

depending strongly continuously on t . Therefore, according to Stone's theorem, we may define the Hamiltonian $\hat{H}_{(0)}(t)$ by³

$$\hat{H}_{(0)}(t) \stackrel{\text{def}}{=} \left(i \frac{d}{dt} \left(\hat{U}_{(0)}(t)^{-1} \right) \right) \hat{U}_{(0)}(t) = -\hat{U}_{(0)}(t)^{-1} i \frac{d}{dt} \hat{U}_{(0)}(t),$$

even if we do not have homogeneity in time. This implies

$$\begin{aligned} i\partial_t \hat{\Omega}(t, t_-) &= \hat{H}_I(t) \hat{\Omega}(t, t_-), \text{ where:} \\ \hat{H}_I(t) &\stackrel{\text{def}}{=} \hat{U}_0(t) \left(\hat{H}(t) - \hat{H}_0(t) \right) \hat{U}_0(t)^{-1}, \end{aligned} \quad (3.9)$$

which, formally, is equivalent to⁴

$$\hat{\Omega}(t, t_-) = \hat{1} - i \int_{t_-}^t \hat{H}_I(t_+) \hat{\Omega}(t_+, t_-) dt_+, \quad (3.10)$$

since $\hat{\Omega}(t_-, t_-) = \hat{1}$, by (3.7). As usual, this integral equation may be formally solved by iteration giving the so-called **DYSON series**.

3.1.2 Canonical Field Quantization

Field Equations

Hoping to convert the free KLEIN-GORDON theory into a model with nontrivial S -matrix, one adds a **local self-interaction** term of the form⁵ $\lambda_b \mathop{:}F(\hat{\Phi}(x))\mathop{:}$ with **coupling constant** $\lambda_b > 0$ as **perturbation** to the KLEIN-GORDON equation.⁶

$$(\square + m_b^2) \hat{\Phi}(x) = -\lambda_b \mathop{:}F(\hat{\Phi}(x))\mathop{:}. \quad (3.11)$$

Best studied is the so-called $\lambda(\Phi^4)_4$ -theory given formally by $F(\Phi) = 2\Phi^3$. Nevertheless, nobody succeeded up to now in giving this theory a precise meaning by rigorous construction. This is due to tremendous technical difficulties connected with 4-dimensionality of physical space-time.⁷ In 2- or 3-dimensional model space-time these difficulties are much less severe and have already been overcome (See

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³Note that $\left(\frac{d}{dt} \hat{U}^{-1} \right) \hat{U} + \hat{U}^{-1} \frac{d}{dt} \hat{U} = 0$.

⁴In naive quantum electrodynamics (before renormalization) one has:

$$\hat{H}_I(t) = \int_{x^0=t} g_{\mu\nu} \hat{j}_{\text{free}}^\mu(x) \hat{A}_{\text{free}}^\nu(x) d\mathbf{x}.$$

⁵Locality, i.e. dependence of the interaction term on the field values at the space-time point x , corresponds to the **point particle** picture.

⁶For the corresponding **classical** field theory see (Reed, 1976). For the problem of defining the operator function $F(\hat{\Phi}(x))$ via $\mathop{:}\mathop{:}$ (in the sense of (3.20)) see (Segal, 1962; Segal, 1983). Negative values of m_b^2 lead to *spontaneous symmetry breaking*.

⁷Actually there are even indications that such a construction is not possible; see (Bég and Furlong, 1985) and references given there.

e.g. (Glimm and Jaffe, 1981; Constantinescu, 1980) and (Streater and Wightman, 1989, Appendix).

Time-Zero Fields

Let us assume that there is a rigorous construction for the interacting theory formally described above. Moreover assume – in spite of all knowledge to the contrary – that the interaction picture, as described in Sect. 3.1.1, is applicable to this theory with the corresponding “free” system being described by the neutral scalar field $\hat{\Phi}_0(x)$ with *physical mass*⁸ m :

$$\begin{aligned}\hat{\Phi}(0, \mathbf{x}) &= \hat{\Phi}_0(0, \mathbf{x}), \\ \hat{\Pi}(0, \mathbf{x}) &\stackrel{\text{def}}{=} \partial_0 \hat{\Phi}(x)|_{x^0=0} = \hat{\Pi}_0(0, \mathbf{x}) \stackrel{\text{def}}{=} \partial_0 \hat{\Phi}_0(x)|_{x^0=0}.\end{aligned}\quad (3.12)$$

Under these conditions

$$:G(\hat{\Phi}(x), \hat{\Pi}(x)): = \hat{U}(x^0):G\left(\hat{\Phi}_0(0, \mathbf{x}), \hat{\Pi}_0(0, \mathbf{x})\right):\hat{U}(x^0)^{-1}\quad (3.13)$$

might be a good definition⁹ for well-behaved functionals G of $\Phi(x)$ and $\Pi(x)$. Here $::$ denotes normal ordering, i.e. the factors of monomials have to be interchanged – as if they were commuting – such that no creation operator is on the right of any annihilation operator.

In any case, by (2.39), (2.40), and (2.43), we have

$$\hat{\Phi}_0(x) = (2\pi)^{-3/2} \int_{p^0=\omega_{\mathbf{p}}} \frac{d\mathbf{p}}{2p^0} \left(\hat{a}(\mathbf{p})e^{-ip^0x^0} + \hat{a}^*(-\mathbf{p})e^{+ip^0x^0} \right) e^{+i\mathbf{p}\mathbf{x}}\quad (3.14)$$

and hence **for real-valued** $\varphi \in \mathcal{S}(\mathbb{R}^3)$:

$$\begin{aligned}\int d\mathbf{x} \hat{\Phi}_0(0, \mathbf{x})\varphi(\mathbf{x}) &= \hat{A}_{\tilde{\varphi}} + \hat{A}_{\tilde{\varphi}}^*, & \int d\mathbf{x} \hat{\Pi}_0(0, \mathbf{x})\varphi(\mathbf{x}) &= -i\hat{A}_{2\omega\tilde{\varphi}} + i\hat{A}_{2\omega\tilde{\varphi}}^*, \\ \text{where: } \tilde{\varphi}(\mathbf{p}) &\stackrel{\text{def}}{=} (2\pi)^{-3/2} \int d\mathbf{x} \varphi(\mathbf{x})e^{-i\mathbf{p}\mathbf{x}}, & \hat{A}_{\tilde{f}} &\stackrel{\text{def}}{=} \int \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}} \hat{a}(\mathbf{p})\tilde{f}(\mathbf{p}).\end{aligned}\quad (3.15)$$

By (3.14) and (2.34), (3.12) implies the *canonical commutation relations*

$$\boxed{\begin{aligned}\left[\hat{\Phi}(0, \mathbf{x}), \hat{\Pi}(0, \mathbf{y}) \right]_- &= i\delta(\mathbf{x} - \mathbf{y}), \\ \left[\hat{\Phi}(0, \mathbf{x}), \hat{\Phi}(0, \mathbf{y}) \right]_- &= \left[\hat{\Pi}(0, \mathbf{x}), \hat{\Pi}(0, \mathbf{y}) \right]_- = 0\end{aligned}}\quad (3.16)$$

on D_0 , as defined by (2.27).

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⁸In case m coincided with the *bare mass* m_b there would be no chance for $e^{i\hat{H}t} = \hat{V}_{\text{out}}e^{i\hat{H}_0t}\hat{V}_{\text{out}}^{-1}$ to hold together with (3.21), since the latter implies coincidence of the spectra of \hat{H} and \hat{H}_0 .

⁹Note, however, that $:G\left(\hat{\Phi}_0(0, \mathbf{x}), \hat{\Pi}_0(0, \mathbf{x})\right):$ is well defined as quadratic form on D_0 (see, e.g., (Reed und Simon, 1972, Sect. VIII.6) for the definition of quadratic forms) but not necessarily as $L(D_0, \mathcal{H})$ -valued generalized function of \mathbf{x} .

Let $\{\check{f}_\mu\}_\mu$ be a complete orthonormal system of scalar 1-particle wave functions fulfilling

$$\check{f}_\mu(\mathbf{p}) = \overline{\check{f}_\mu(-\mathbf{p})} \quad (3.17)$$

and define

$$\varphi_\mu(\mathbf{x}) \stackrel{\text{def}}{=} (2\pi)^{-3/2} \int d\mathbf{p} \check{f}_\mu(\mathbf{p}) e^{i\mathbf{p}\mathbf{x}}, \quad \psi_\mu(\mathbf{x}) \stackrel{\text{def}}{=} (2\pi)^{-3/2} \int d\mathbf{p} \frac{\check{f}_\mu(\mathbf{p})}{4\omega_{\mathbf{p}}} e^{i\mathbf{p}\mathbf{x}}. \quad (3.18)$$

Then, by (3.12)/(3.15) and in agreement with (3.16),

$$\begin{aligned} \pi \left(\hat{U}_\nu(\tau) \right) &\stackrel{\text{def}}{=} \exp \left(i \int d\mathbf{x} \hat{\Pi}(0, \mathbf{x}) \tau \psi_\nu(\mathbf{x}) \right), \\ \pi \left(\hat{V}_\mu(\tau) \right) &\stackrel{\text{def}}{=} \exp \left(i \int d\mathbf{x} \hat{\Phi}(0, \mathbf{x}) \tau \varphi_\mu(\mathbf{x}) \right), \end{aligned} \quad (3.19)$$

defines a regular representation¹⁰ of the WEYL commutation relations (1.45):

$$\hat{U}_\nu(\tau) \hat{V}_\mu(s) = e^{i\tau s \delta_{\nu\mu}} \hat{V}_\mu(s) \hat{U}_\nu(\tau) \quad \text{etc.}$$

Even if (3.12) is not fulfilled, the field theory is called *canonical*, whenever (3.19) is a representation of the WEYL commutation relations. Then π has a unique extension to a true representation of the CCR algebra.

Exercise 44 Show that, provided (3.12) holds, the representation of the CCR algebra resulting from (3.19) is equivalent to the FOCK representation (discussed in Sect. 1.3.3).¹¹

Exercise 45 Using (3.15), show the following:

- (i) The (identical representation of the) C^* -algebra (in \mathcal{H}_0) generated by the bounded functions of smeared time-zero fields $\hat{\Phi}_0(0, \mathbf{x})$, $\hat{\Pi}_0(0, \mathbf{x})$ is **irreducible**.
- (ii) Ω_0 is **cyclic** w.r.t. (the restriction of this representation to) the abelian subalgebra generated by bounded functions of the smeared field $\hat{\Phi}_0(0, \mathbf{x})$.

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¹⁰Note the nontrivial dependence on the mass value m .

¹¹For comparison with the theory of ordinary independent quantum oscillators, note that

$$\pi(\hat{a}_\nu) \stackrel{\text{def}}{=} \frac{1}{2} \pi(\hat{x}_\nu) + i\pi(\hat{p}_\nu) = \frac{\sqrt{mD}\pi(\hat{x}_\nu) + i\hat{p}_\nu}{\sqrt{2m}} \quad \text{for } m = D = \frac{1}{2}.$$

Use of the Canonical Commutation Relations

If the canonical commutation relations hold and if $:::$ is defined such that¹²

$$\begin{aligned} & [:\hat{A}(\mathbf{x}')^N:; \hat{B}(\mathbf{x})]_- \\ &= \begin{cases} :N\hat{A}(\mathbf{x}')^{N-1}[\hat{A}(\mathbf{x}'), \hat{B}(\mathbf{x})]_-: & \text{if } \{\hat{A}(\cdot), \hat{B}(\cdot)\} = \{\hat{\Phi}(0, \cdot), \hat{\Pi}(0, \cdot)\} \\ & \text{or } \{\hat{A}(\cdot), \hat{B}(\cdot)\} = \{\partial_j \hat{\Phi}(0, \cdot), \hat{\Pi}(0, \cdot)\} \\ 0 & \text{if } \hat{A}(\cdot) \in \{\hat{\Phi}(0, \cdot), \partial_j \hat{\Phi}(0, \cdot)\} \ni \hat{B}(\cdot) \\ & \text{or } \hat{A}(\cdot) = \hat{B}(\cdot) = \hat{\Pi}(0, \cdot), \end{cases} \quad (3.20) \end{aligned}$$

holds for $j \in \{1, 2, 3\}$ we get¹³ a formal solution of the field equation (3.11) in the form

$$\begin{aligned} & \hat{\Phi}(t, \mathbf{x}) = e^{i\hat{H}t} \hat{\Phi}(0, \mathbf{x}) e^{-i\hat{H}t}, \text{ where:} \\ & \hat{H} \stackrel{\text{def}}{=} \frac{1}{2} \int : \left(\left(\hat{\Pi}(0, \mathbf{x}) \right)^2 + \left(\nabla \hat{\Phi}(0, \mathbf{x}) \right)^2 + m_b^2 \left(\hat{\Phi}(0, \mathbf{x}) \right)^2 + 2\lambda_b U \left(\hat{\Phi}(0, \mathbf{x}) \right) \right) : d\mathbf{x}, \\ & \quad + \text{const.} \\ & U \stackrel{\text{def}}{=} \text{indefinite integral of } F, \end{aligned}$$

(3.21)

which – in a cutoff version¹⁴ – was the original starting point for *constructive field theory*.

Reminder: In classical LAGRANGE field theory the Hamiltonian

$$H(\Phi_t, \Pi_t) = \frac{1}{2} \int (\Pi_t(\mathbf{x})^2 + (\nabla \Phi_t(\mathbf{x}))^2 + m_b^2 \Phi_t(\mathbf{x})^2 + 2\lambda_b U(\Phi_t(\mathbf{x}))) d\mathbf{x},$$

corresponds to the Lagrangian

$$L(\Phi_t, \dot{\Phi}_t) = \frac{1}{2} \int \left(\left(\dot{\Phi}_t(\mathbf{x}) \right)^2 - (\nabla \Phi_t(\mathbf{x}))^2 - m_b^2 \Phi_t(\mathbf{x})^2 - 2\lambda_b U(\Phi_t(\mathbf{x})) \right) d\mathbf{x}$$

with

$$\Pi_t(\mathbf{x}) \stackrel{\text{def}}{=} \frac{\delta L}{\delta \dot{\Phi}_t(\mathbf{x})} = \dot{\Phi}_t(\mathbf{x}).$$

The EULER-LAGRANGE equation

$$\frac{d}{dt} \frac{\delta L}{\delta \dot{\Phi}_t(\mathbf{x})} - \frac{\delta L}{\delta \Phi_t(\mathbf{x})} = 0$$

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¹²Note that, for bounded operators, $[\hat{A}\hat{B}, \hat{C}]_- = \hat{A}[\hat{B}, \hat{C}]_- + [\hat{A}, \hat{C}]_- \hat{B}$ and, consequently:

$$[\hat{A}, \hat{B}]_- \sim \hat{1} \implies [\hat{A}^N, \hat{B}]_- = N\hat{A}^{N-1}[\hat{A}, \hat{B}]_-.$$

For (3.12)/(3.13), (3.20) is (formally) a consequence of WICK's theorem (Theorem 3.2.1).

¹³Thanks to

$$\begin{aligned} [:\hat{A}(\mathbf{x}')^N:; \hat{\Pi}(0, \mathbf{x})]_- &= i\delta(\mathbf{x} - \mathbf{x}') : \frac{\partial}{\partial \hat{\Phi}(0, \mathbf{x}')} \hat{A}(\mathbf{x}')^N :, \\ [:\hat{A}(\mathbf{x}')^N:; \hat{\Phi}(0, \mathbf{x})]_- &= -i\delta(\mathbf{x} - \mathbf{x}') : \frac{\partial}{\partial \hat{\Pi}(0, \mathbf{x}')} \hat{A}(\mathbf{x}')^N :, \end{aligned}$$

¹⁴Note that, e.g., even $:\hat{\Phi}_0(0, \mathbf{x})^4:$ is not well-defined (except on 2-dimensional space-time).

(more precisely: family of equations, parameterized by \mathbf{x}) for this Lagrangian is the field equation

$$(\square + m_b^2) \Phi(x) = -\lambda_b F(\Phi(x)), \quad \Phi(t, \mathbf{x}) \stackrel{\text{def}}{=} \hat{\Phi}_t(\mathbf{x}).$$

Note that the EULER-LAGRANGE equation is equivalent to the Hamilton equations

$$\begin{aligned} \dot{\hat{\Phi}}_t(\mathbf{x}) &= -\{H(\hat{\Phi}_t, \Pi_t), \hat{\Phi}_t(\mathbf{x})\} = \frac{\delta H}{\delta \Pi_t(\mathbf{x})}, \\ \dot{\Pi}_t(\mathbf{x}) &= -\{H(\hat{\Phi}_t, \Pi_t), \Pi_t(\mathbf{x})\} = -\frac{\delta H}{\delta \hat{\Phi}_t(\mathbf{x})}, \end{aligned}$$

where the POISSON **bracket** $\{, \}$ is defined by

$$\{F, G\} \stackrel{\text{def}}{=} \int \left(\frac{\delta F}{\delta \hat{\Phi}_t(\mathbf{x}')} \frac{\delta G}{\delta \Pi_t(\mathbf{x}')} - \frac{\delta F}{\delta \Pi_t(\mathbf{x}')} \frac{\delta G}{\delta \hat{\Phi}_t(\mathbf{x}')} \right) d\mathbf{x}'$$

for functionals F, G of $\hat{\Phi}_t, \Pi_t$.

Indeed,¹⁵

$$\begin{aligned} \dot{\hat{\Phi}}(t, \mathbf{x}) &= e^{i\hat{H}t} [i\hat{H}, \hat{\Phi}(0, \mathbf{x})]_- e^{-i\hat{H}t} \\ &\stackrel{(3.21)}{=} e^{i\hat{H}t} \frac{i}{2} \int [:\Pi(0, \mathbf{x}')^2:, \hat{\Phi}(0, \mathbf{x})]_- d\mathbf{x}' e^{-i\hat{H}t} \\ &\stackrel{(3.13), (3.16)}{=} e^{i\hat{H}t} i \int :\Pi(0, \mathbf{x}')[\Pi(0, \mathbf{x}'), \hat{\Phi}(0, \mathbf{x})]_-: d\mathbf{x}' e^{-i\hat{H}t} \\ &\stackrel{(3.20)}{=} e^{i\hat{H}t} i \int :\Pi(0, \mathbf{x}')[\Pi(0, \mathbf{x}'), \hat{\Phi}(0, \mathbf{x})]_-: d\mathbf{x}' e^{-i\hat{H}t} \\ &\stackrel{(3.16)}{=} e^{i\hat{H}t} \Pi(0, \mathbf{x}) e^{-i\hat{H}t} \end{aligned} \tag{3.22}$$

and a similar **formal** calculation verifies

$$\ddot{\hat{\Phi}}(t, \mathbf{x}) \stackrel{(3.22)}{=} e^{i\hat{H}t} [i\hat{H}, \Pi(0, \mathbf{x})]_- e^{-i\hat{H}t}$$

to be in agreement with the field equation (3.11).

However, locality and relativistic covariance of the formal solution (3.21) are not so easy to establish.¹⁶

Exercise 46 Show for arbitrary $\varphi \in \mathcal{S}(\mathbb{R}^3)$ that

$$\int \frac{|\tilde{\varphi}(\mathbf{p}_1 + \mathbf{p}_2)|^2}{(\omega_{\mathbf{p}_1} \omega_{\mathbf{p}_2})^3} d\mathbf{p}_1 d\mathbf{p}_2 < \infty$$

and therefore

$$\int \left(\hat{\Pi}(0, \mathbf{x}) \right)^2 + \left(\nabla \hat{\Phi}(0, \mathbf{x}) \right)^2 + m_b^2 \left(\hat{\Phi}(0, \mathbf{x}) \right)^2 : \varphi(\mathbf{x}) d\mathbf{x} \in L(D_0, \mathcal{H}).$$

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¹⁵Note that we do not care about the domain of definition for \hat{H} . For $:F(\hat{\Phi}_0(0, \mathbf{x})):\Omega_0 \neq 0$, the case of interest, there are obvious difficulties with the FOCK representation.

¹⁶Note that $[\hat{A}, \hat{B}]_- = 0 \not\stackrel{\text{i.g.}}{\Rightarrow} [e^{i\hat{A}}, e^{i\hat{B}}]_- = 0$ (see e.g. (Reed und Simon, 1972, Vol. I, Sect. VIII.5) and (Fröhlich, 1977)).

Inapplicability of the FOCK Representation

Let us assume (3.12) to be valid. Then

$$\begin{aligned}\hat{U}(0, \mathbf{a})\Phi_0(0, \mathbf{x})\hat{U}(0, \mathbf{a})^{-1} &= \hat{U}(0, \mathbf{a})\Phi(0, \mathbf{x})\hat{U}(0, \mathbf{a})^{-1} \\ &= \Phi(0, \mathbf{x} + \mathbf{a}) \\ &= \Phi_0(0, \mathbf{x} + \mathbf{a}) \\ &= \hat{U}_0(0, \mathbf{a})\Phi_0(0, \mathbf{x})\hat{U}_0(0, \mathbf{a})^{-1}\end{aligned}$$

and the corresponding conclusion for $\Pi_0(0, \mathbf{a})$ imply (compare Exercise 45(i)):

$$\hat{U}(0, \mathbf{a}) = e^{i\varphi(\mathbf{a})}\hat{U}_0(0, \mathbf{a})$$

for suitable real-valued $\varphi(\mathbf{a})$. There is only one 1-dimensional subspace which is invariant under all $\hat{U}_0(0, \mathbf{a})$, and this contains the FOCK vacuum Ω_0 , characterized up to a factor by $\hat{\mathbf{P}}_0\Omega_0 = 0$. For the *physical vacuum* state vector Ω , characterized up to a factor by

$$\hat{U}(a)\Omega = \Omega \quad \forall a \in \mathbb{R}^4,$$

we have

$$\hat{U}_0(0, \mathbf{a})\Omega = e^{-i\varphi(\mathbf{a})}\hat{U}(0, \mathbf{a})\Omega = e^{-i\varphi(\mathbf{a})}\Omega.$$

Therefore also Ω is an element of the invariant 1-dimensional subspace, i.e.:

$$\Omega = e^{i\alpha}\Omega_0.$$

This means that both $\hat{U}(0, \mathbf{a})$ and $\hat{U}_0(0, \mathbf{a})$ leave Ω_0 (and Ω) invariant and thus have to coincide:

$$\hat{U}_0(0, \mathbf{a}) = \hat{U}(0, \mathbf{a}) \quad \forall \mathbf{a} \in \mathbb{R}^3.$$

Since, for obvious physical reasons, (3.8) should be supplemented by

$$\hat{U}(0, \mathbf{a}) = \hat{V}_{\text{out}}\hat{U}_0(0, \mathbf{a})\hat{V}_{\text{out}}^{-1} \quad \forall \mathbf{a} \in \mathbb{R}^3,$$

we get commutativity of \hat{V}_{out} with all $\hat{U}_0(0, \mathbf{a})$. Now, since \hat{P}_0^0 is a function of $\hat{\mathbf{P}}_0$, commutativity with all $\hat{U}_0(0, \mathbf{a})$ implies commutativity with all $\hat{U}_0(t)$. Thus, by (3.8) we have

$$\hat{U}_0(t, 0) = \hat{U}(t, 0) \quad \forall t \in \mathbb{R}$$

and hence

$$\hat{\Phi}_0(x) = \hat{\Phi}(x) \quad \forall x \in \mathbb{R}^4.$$

HAAG'S *theorem*¹⁷ says that this conclusion is correct even without (3.8) and its generalization and without specification of \hat{H} , if the theory fulfills the WIGHTMAN

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¹⁷A rigorous proof is given in (Streater and Wightman, 1989, Sect. 4.5). In a first step, relativistic covariance is used to show that

$$\langle \Omega | \hat{\Phi}(x)\hat{\Phi}(y)\Omega \rangle = \langle \Omega_0 | \hat{\Phi}_0(x)\hat{\Phi}_0(y)\Omega_0 \rangle \quad (*)$$

holds for $x \times y$, since the equations holds for $x^0 = y^0 = 0$. This together with the spectrum condition implies that (*) holds for all $x, y \in \mathbb{R}^4$, as can be shown by standard techniques of axiomatic field theory (for a stronger result see (Lücke, 1979, Corollary)). Then the JOST-SCHROER theorem (Theorem 2.2.18) says that (*) can only hold for all $x, y \in \mathbb{R}^4$ if $\hat{\Phi}_0(x)$ and $\hat{\Phi}(x)$ are unitarily equivalent.

axioms formulated in 2.2.1.

So, unfortunately, it is not possible to define the Hamiltonian on a suitable domain to make the formal solution (3.21) of (3.11) a true one – as long as one insists in (3.12) for **all** $\mathbf{x} \in \mathbb{R}^3$.

In spite of HAAG's theorem there is still hope (compare (Baumann,)) that a nontrivial **canonical** $\lambda(\Phi^4)_4$ -theory might exist.¹⁸ For such a quantum field theory, of course, the representation π given by (3.19) must be **inequivalent** to the FOCK representation. All this was illustrated in *constructive field theory* by several models living on space-time of dimension < 4 .

For the free field, of course, nothing is wrong with the **canonical quantization** procedure (3.21) (for $\lambda_b = 0$).

3.2 Canonical Perturbation Theory

3.2.1 DYSON Series and WICK's Theorem

Let us consider an IS for which the interaction picture works as described in Section 3.1.1:

$$\begin{array}{rcl} \hat{S}_0 & \stackrel{(3.7)}{=} & \lim_{t_{\pm} \rightarrow \pm\infty} \hat{\Omega}(t_+, t_-), \\ \hat{\Omega}(t_+, t_-) & \stackrel{(3.10)}{=} & \hat{1} - i \int_{t_-}^{t_+} \hat{H}_I(t) \hat{\Omega}(t, t_-) dt, \\ \hat{H}_I(t) & \stackrel{(3.9)}{=} & \hat{U}_0(t) \left(\hat{H}(t) - \hat{H}_0(t) \right) \hat{U}_0(t)^{-1}. \end{array} \quad (3.23)$$

Let us assume

$$\hat{H}_I(t) = \lambda \hat{\hat{H}}_I(t)$$

and that – on a suitable domain – $\hat{\Omega}(t, t_-)$ depends sufficiently smoothly on λ . Then the LEIBNIZ rule gives

$$\partial_\lambda^n \hat{\Omega}(t, t_-) = -i \int_{t_-}^t \sum_{\nu=0}^n \binom{n}{\nu} \left(\partial_\lambda^\nu \hat{H}_I(t') \right) \partial_\lambda^{n-\nu} \hat{\Omega}(t', t_-) dt' \quad \text{for } n > 0$$

and, because of

$$\left(\partial_\lambda^\nu \hat{H}_I(t) \right)_{|\lambda=0} = \begin{cases} \hat{\hat{H}}_I(t) & \text{for } \nu = 1, \\ 0 & \text{else,} \end{cases}$$

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¹⁸Note that existence of a unitary operator \hat{U}_R fulfilling the conditions

$$\begin{aligned} \hat{\Phi}(0, \mathbf{x}) &= \hat{U}_R \hat{\Phi}_0(0, \mathbf{x}) \hat{U}_R^{-1}, \quad \hat{\Pi}(0, \mathbf{x}) = \hat{U}_R \hat{\Pi}_0(0, \mathbf{x}) \hat{U}_R^{-1}, \\ :F(\hat{\Phi}(x)) : &= \hat{U}(x^0) \hat{U}_R :F(\hat{\Phi}_0(0, \mathbf{x})) : \hat{U}_R^{-1} \hat{U}(x^0)^{-1} \end{aligned}$$

for $|x| < R$ would have been sufficient for the formal proof of (3.11) in this region.

the iteration formula

$$\partial_\lambda^n \hat{\Omega}(t, t_-)|_{\lambda=0} = -i \int_{t_-}^t n \hat{H}_I(t') \left(\partial_\lambda^{n-1} \hat{\Omega}(t', t_-) \right) |_{\lambda=0} dt'.$$

Since,

$$\hat{\Omega}(t, t_-)|_{\lambda=0} = \hat{1},$$

this gives the (not necessarily converging) Taylor expansion

$$\begin{aligned} \hat{\Omega}(t_+, t_-) &= \hat{1} + \sum_{n=1}^{\infty} (-i)^n \underbrace{\int_{t_- < t_1 < \dots < t_n < t_+} \hat{H}_I(t_n) \cdots \hat{H}_I(t_1) dt_1 \cdots dt_n}_{\frac{1}{n!} \sum_{\pi \in S_n} \int_{t_- < t_{\pi 1} < \dots < t_{\pi n} < t_+} T(\hat{H}_I(t_n) \cdots \hat{H}_I(t_1)) dt_1 \cdots dt_n} \\ &= \hat{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t_-}^{t_+} \cdots \int_{t_-}^{t_+} T(\hat{H}_I(t_n) \cdots \hat{H}_I(t_1)) dt_1 \cdots dt_n, \end{aligned}$$

where¹⁹

$$T(\hat{H}_I(t_n) \cdots \hat{H}_I(t_1)) \stackrel{\text{def}}{=} \hat{H}_I(t_{\pi n}) \cdots \hat{H}_I(t_{\pi 1}) \quad \text{for } \pi \in S_n \text{ with } t_{\pi 1} < \dots < t_{\pi n}$$

is the so-called **chronological product** of $\hat{H}_I(t_n), \dots, \hat{H}_I(t_1)$. The usual shorthand notation for the resulting formal perturbation expansion of the S -matrix is:

$$\boxed{\hat{S}_0 = T \exp \left(-i \int_{-\infty}^{+\infty} \hat{H}_I(t) dt \right)}. \quad (3.24)$$

In view of $\lambda\Phi_4^4$ -theory let us formally assume

$$\boxed{\hat{H}_I(x^0) = i \int g(x) \hat{S}_1(x) dx}, \quad (3.25)$$

where $\hat{S}_1(x)$ is a normal ordered function of the **free** field $\hat{\Phi}_0(x)$ and its derivatives at the space-time point x . Then (3.24) becomes:

$$\boxed{\begin{aligned} \hat{S}_0 &= \hat{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int \cdots \int \hat{S}_n(x_1, \dots, x_n) g(x_1) \cdots g(x_n) dx_1 \cdots dx_n, \\ \text{where } \hat{S}_n(x_1, \dots, x_n) &\stackrel{\text{def}}{=} T(\hat{S}_1(x_1) \cdots \hat{S}_1(x_n)). \end{aligned}} \quad (3.26)$$

In order to facilitate evaluation of the S -matrix elements the chronological products should be expressed by normal ordered products. Formally this may be done by applying WICK's theorem.

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¹⁹This is the definition for BOSE fields, only. However, even if the theory contained FERMI fields the Hamiltonian ought to be of BOSE type.

Theorem 3.2.1 (WICK's theorem) *Let*²⁰

$$\hat{\chi}_\nu(x) \in \left\{ \hat{\Phi}_0(x), \partial_0 \hat{\Phi}_0(x), \dots, \partial_3 \hat{\Phi}_0(x) \right\} \quad \text{for } \nu = 1, \dots, j_N$$

and let $j_1 < j_2 < \dots < j_N$. Then

$$\begin{aligned} & : \hat{\chi}_1(x_1) \cdots \hat{\chi}_{j_1}(x_{j_1}) :: \hat{\chi}_{j_1+1}(x_{j_1+1}) \cdots \hat{\chi}_{j_2}(x_{j_2}) : \cdots : \hat{\chi}_{j_{N-1}+1}(x_{j_{N-1}+1}) \cdots \hat{\chi}_{j_N}(x_{j_N}) : \\ & =: \exp \left(\sum_{\substack{\nu \leq j_k < \mu \\ \text{for suit. } k}} \int \int dx dy \langle \Omega_0 | \hat{\chi}_\nu(x) \hat{\chi}_\mu(y) \Omega_0 \rangle \frac{\delta}{\delta \hat{\chi}_\nu(x)} \frac{\delta}{\delta \hat{\chi}_\mu(y)} \right) \\ & \qquad \qquad \qquad \hat{\chi}_1(x_1) \cdots \hat{\chi}_{j_N}(x_{j_N}) :, \end{aligned}$$

if – formally – the fields $\hat{\chi}_\nu(x)$ are considered as independent functional variables.

Proof: From the simple chain of equations

$$\begin{aligned} & \hat{\Phi}_0^+(x_k) \cdots \hat{\Phi}_0^+(x_n) \hat{\Phi}_0^-(x_{n+1}) \\ & = \hat{\Phi}_0^+(x_k) \cdots \hat{\Phi}_0^+(x_{n-1}) \hat{\Phi}_0^-(x_{n+1}) \hat{\Phi}_0^+(x_n) \\ & \quad + \hat{\Phi}_0^+(x_k) \cdots \hat{\Phi}_0^+(x_{n-1}) [\hat{\Phi}_0^+(x_n), \hat{\Phi}_0^-(x_{n+1})]_- \\ & = \hat{\Phi}_0^+(x_k) \cdots \hat{\Phi}_0^+(x_{n-2}) \hat{\Phi}_0^-(x_{n+1}) \hat{\Phi}_0^+(x_{n-1}) \hat{\Phi}_0^+(x_n) \\ & \quad + \hat{\Phi}_0^+(x_k) \cdots \hat{\Phi}_0^+(x_{n-2}) [\hat{\Phi}_0^+(x_{n-1}), \hat{\Phi}_0^-(x_{n+1})]_- \hat{\Phi}_0^+(x_n) \\ & \quad + \hat{\Phi}_0^+(x_k) \cdots \hat{\Phi}_0^+(x_{n-1}) [\hat{\Phi}_0^+(x_n), \hat{\Phi}_0^-(x_{n+1})]_- \\ & \quad \vdots \\ & = \hat{\Phi}_0^-(x_{n+1}) \hat{\Phi}_0^+(x_k) \cdots \hat{\Phi}_0^+(x_n) + \sum_{\nu=k}^n [\hat{\Phi}_0^+(x_\nu), \hat{\Phi}_0^-(x_{n+1})]_- \prod_{\substack{\mu=k \\ \mu \neq \nu}}^n \hat{\Phi}_0^+(x_\mu) \end{aligned}$$

we easily conclude that

$$\begin{aligned} & \hat{\Phi}_0^-(x_1) \cdots \hat{\Phi}_0^-(x_{k-1}) \hat{\Phi}_0^+(x_k) \cdots \hat{\Phi}_0^+(x_n) \hat{\chi}_{n+1}(x_{n+1}) \\ & =: \hat{\Phi}_0^-(x_1) \cdots \hat{\Phi}_0^-(x_{k-1}) \hat{\Phi}_0^+(x_k) \cdots \hat{\Phi}_0^+(x_n) \hat{\chi}_{n+1}(x_{n+1}) : \\ & \quad + \sum_{\nu=k}^n [\hat{\Phi}_0^+(x_\nu), \hat{\chi}_{n+1}(x_{n+1})]_- \hat{\Phi}_0^-(x_1) \cdots \hat{\Phi}_0^-(x_{k-1}) \prod_{\substack{\mu=k \\ \mu \neq \nu}}^n \hat{\Phi}_0^+(x_\mu) \end{aligned}$$

holds for $\hat{\chi}_{n+1}(x) \in \left\{ \hat{\Phi}_0^-(x), \hat{\Phi}_0^+(x) \right\}$. This implies

$$\begin{aligned} & : \hat{\chi}_1(x_1) \cdots \hat{\chi}_n(x_n) : \hat{\chi}_{n+1}(x_{n+1}) \\ & =: \hat{\chi}_1(x_1) \cdots \hat{\chi}_{n+1}(x_{n+1}) : + \sum_{\nu=1}^n \langle \Omega_0 | \hat{\chi}_\nu(x_\nu) \hat{\chi}_{n+1}(x_{n+1}) \Omega_0 \rangle : \prod_{\substack{\mu=1 \\ \mu \neq \nu}}^n \hat{\chi}_\mu(x_\mu) : \end{aligned}$$

or, written in a suggestive formal way,

$$\begin{aligned} & : \hat{\chi}_1(x_1) \cdots \hat{\chi}_n(x_n) : \hat{\chi}_{n+1}(x_{n+1}) \\ & =: \exp \left(\sum_{\nu=1}^n \int \int dx dy \langle \Omega_0 | \hat{\chi}_\nu(x) \hat{\chi}_{n+1}(y) \Omega_0 \rangle \frac{\delta}{\delta \hat{\chi}_\nu(x)} \frac{\delta}{\delta \hat{\chi}_{n+1}(y)} \right) \\ & \qquad \qquad \qquad \hat{\chi}_1(x_1) \cdots \hat{\chi}_{n+1}(x_{n+1}) : \end{aligned}$$

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²⁰Here, the $\hat{\chi}_\nu$ could also be partial derivatives of various BOSE fields. For the more general case, where some of the $\hat{\chi}_\nu(x)$ are FERMI fields, see Chapter 4.

for

$$\hat{\chi}_1(x_1), \dots, \hat{\chi}_{n+1}(x_{n+1}) \in \left\{ \hat{\Phi}_0^+(x), \hat{\Phi}_0^-(x) \right\}. \quad (*)$$

Induction w.r.t. n , finally, gives

$$\begin{aligned} & \hat{\chi}_1(x_1) \cdots \hat{\chi}_n(x_n) \hat{\chi}_{n+1}(x_{n+1}) \\ =: & \exp \left(\sum_{1 \leq \nu < \mu \leq n+1} \int \int dx dy \langle \Omega_0 | \hat{\chi}_\nu(x) \hat{\chi}_\mu(y) \Omega_0 \rangle \frac{\delta}{\delta \hat{\chi}_\nu(x)} \frac{\delta}{\delta \hat{\chi}_\mu(y)} \right) \\ & \hat{\chi}_1(x_1) \cdots \hat{\chi}_{n+1}(x_{n+1}) :, \end{aligned}$$

provided $(*)$ holds. From this the statement of the theorem follows easily for

$$\hat{\chi}_1(x) = \dots = \hat{\chi}_{n+1}(x) = \hat{\Phi}_0(x)$$

and then, by just forming derivatives, the full statement. \blacksquare

Corollary 3.2.2 *Let P_1, \dots, P_n be polynomials. Then²¹*

$$\begin{aligned} & :P_1(\hat{\Phi}_0(x_1)) : \cdots :P_n(\hat{\Phi}_0(x_n)) : \\ =: & \prod_{\nu < \mu} \exp \left(\left\langle \Omega_0 | \hat{\Phi}_0(x_\nu) \hat{\Phi}_0(x_\mu) \Omega_0 \right\rangle \frac{\partial}{\partial \hat{\Phi}_0(x_\nu)} \frac{\partial}{\partial \hat{\Phi}_0(x_\mu)} \right) P_1(\hat{\Phi}_0(x_1)) \cdots P_n(\hat{\Phi}_0(x_n)) : \end{aligned}$$

if – formally – $\hat{\Phi}_0(x_1), \dots, \hat{\Phi}_0(x_n)$ are considered as independent scalar variables.

Exercise 47 Use Corollary 3.2.2 to give a formal proof of²²

$$:e^{\hat{\Phi}_0(x)} : :e^{\hat{\Phi}_0(y)} : = e^{\langle \Omega_0 | \hat{\Phi}_0(x) \hat{\Phi}_0(y) \Omega_0 \rangle} :e^{\hat{\Phi}_0(x) + \hat{\Phi}_0(y)} :$$

and the corresponding formula for the time-ordered product of the two (normal ordered) exponentials.

For simplicity let us assume

$$\hat{S}_1 = :P(\hat{\Phi}_0(x)):$$

for some polynomial P . Then Corollary 3.2.2 implies **formally**

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²¹For the definition of powers of 2-point functions see Equation (3.32) and Exercise 49. Actually, a rigorous proof of Corollary 3.2.2 is not straightforward.

²²For a rigorous definition of expressions of the form $\sum_{n=0}^{\infty} c_n : \hat{\Phi}_0(x)^n :$ with arbitrary $c_n \in \mathbb{R}$ see (Rieckers, 1971).

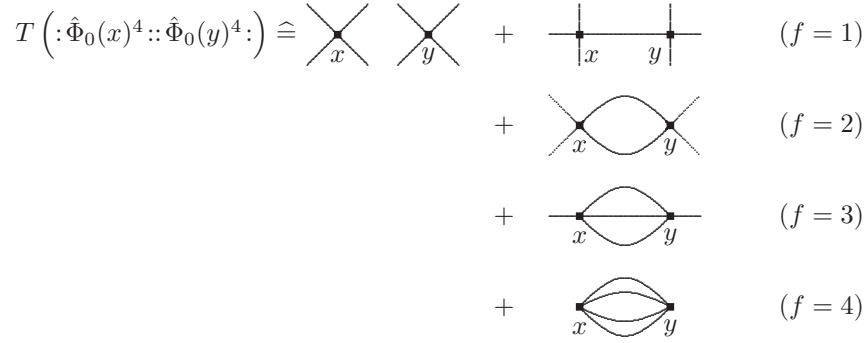


Figure 3.2: WICK's theorem interpreted in terms of diagrams

$$\begin{aligned}
 & T\left(\hat{\Phi}_0(x)^4 :: \hat{\Phi}_0(y)^4\right) \cong \text{diagram} + \text{diagram} \quad (f=1) \\
 & \quad + \text{diagram} \quad (f=2) \\
 & \quad + \text{diagram} \quad (f=3) \\
 & \quad + \text{diagram} \quad (f=4)
 \end{aligned}$$

$$\begin{aligned}
 & T\left(\hat{S}_1(x_1) \cdots \hat{S}_1(x_n)\right) \\
 & = :P\left(\hat{\Phi}_0(x_1)\right) \cdots P\left(\hat{\Phi}_0(x_n)\right): \\
 & \quad + \sum_{\nu < \mu} \sum_{f_{\nu\mu}=1}^{\deg P} :P\left(\hat{\Phi}_0(x_1)\right) \cdots P\left(\hat{\Phi}_0(x_\nu)\right) \cdots P\left(\hat{\Phi}_0(x_\mu)\right) \cdots P\left(\hat{\Phi}_0(x_n)\right): \\
 & \quad + \frac{1}{2} \sum_{\nu_1 < \mu_1} \sum_{\substack{\nu_2 < \mu_2 \\ (\nu_2, \mu_2) \neq (\nu_1, \mu_1)}} \sum_{f_{\nu_1\mu_1}, f_{\nu_2\mu_2}=1}^{\deg P} \overbrace{:P\left(\hat{\Phi}_0(x_1)\right) \cdots}^{f_{\nu\mu}} \\
 & \quad \cdots P\left(\hat{\Phi}_0(x_{\nu_1})\right) \cdots P\left(\hat{\Phi}_0(x_{\nu_2})\right) \cdots P\left(\hat{\Phi}_0(x_{\mu_1})\right) \cdots P\left(\hat{\Phi}_0(x_{\mu_2})\right) \cdots P\left(\hat{\Phi}_0(x_n)\right): \\
 & \quad + \dots
 \end{aligned} \tag{3.27}$$

where

$$\overbrace{\quad \quad \quad}^{f_{\nu\mu}} \cong \frac{1}{f_{\nu\mu}!} \left(i\Delta_F(x_\nu - x_\mu) \frac{\partial}{\partial \hat{\Phi}_0(x_\nu)} \frac{\partial}{\partial \hat{\Phi}_0(x_\mu)} \right)^{f_{\nu\mu}}$$


and

$$\begin{aligned}
 \Delta_F(x-y) & \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow +0} (2\pi)^{-4} \int \frac{1}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} dp \\
 & = -i \langle \Omega_0 | T\left(\hat{\Phi}_0(x)\hat{\Phi}_0(y)\right) \Omega_0 \rangle \quad \text{for } x \neq y.
 \end{aligned} \tag{3.28}$$

This sum over all **contraction** schemes with f -fold contraction lines may be easily identified with a corresponding sum of diagrams.²³

In Fig. 3.2 this is sketched for the special case $P(\xi) = \xi^4$, $n = 2$. Here the

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²³Subdiagrams of the type  (*tadpoles*) do never occur – thanks to WICK ordering of $\hat{S}_1(x)$.

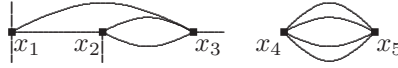
concrete meaning of the sum of diagrams is

$$\begin{aligned}
T\left(:\hat{\Phi}_0(x)^4::\hat{\Phi}_0(y)^4:\right) &= :\hat{\Phi}_0(x)^4\hat{\Phi}_0(y)^4: \\
&+:\left(i\Delta_F(x-y)\frac{\partial}{\partial\hat{\Phi}_0(x)}\frac{\partial}{\partial\hat{\Phi}_0(y)}\right)\hat{\Phi}_0(x)^4\hat{\Phi}_0(y)^4: \\
&+\frac{1}{2}:\left(i\Delta_F(x-y)\frac{\partial}{\partial\hat{\Phi}_0(x)}\frac{\partial}{\partial\hat{\Phi}_0(y)}\right)^2\hat{\Phi}_0(x)^4\hat{\Phi}_0(y)^4: \\
&+\frac{1}{3!}:\left(i\Delta_F(x-y)\frac{\partial}{\partial\hat{\Phi}_0(x)}\frac{\partial}{\partial\hat{\Phi}_0(y)}\right)^3\hat{\Phi}_0(x)^4\hat{\Phi}_0(y)^4: \\
&+\frac{1}{4!}:\left(i\Delta_F(x-y)\frac{\partial}{\partial\hat{\Phi}_0(x)}\frac{\partial}{\partial\hat{\Phi}_0(y)}\right)^4\hat{\Phi}_0(x)^4\hat{\Phi}_0(y)^4:,
\end{aligned}$$

giving the **formal** result

$$\begin{aligned}
T\left(:\hat{\Phi}_0(x)^4::\hat{\Phi}_0(y)^4:\right) &= :\hat{\Phi}_0(x)^4\hat{\Phi}_0(y)^4: \\
&+16i\Delta_F(x-y):\hat{\Phi}_0(x)^3\hat{\Phi}_0(y)^3: \\
&-72\Delta_F(x-y)^2:\hat{\Phi}_0(x)^2\hat{\Phi}_0(y)^2: \\
&-96i\Delta_F(x-y)^3:\hat{\Phi}_0(x)\hat{\Phi}_0(y): \\
&+24\Delta_F(x-y)^4.
\end{aligned} \tag{3.29}$$

Exercise 48 Evaluate



considered as a single diagram (not a sum of diagrams).

Recall that, according to (3.28),

$$\Delta_F(x) = \begin{cases} +\Delta_+(x) & \text{for } x \in \mathbb{R}^4 \setminus \overline{V}_-, \\ -\Delta_-(x) & \text{for } x \in \mathbb{R}^4 \setminus \overline{V}_+, \end{cases}$$

where²⁴

$$\begin{aligned}
\Delta_+(x-y) &\stackrel{\text{def}}{=} -i\langle\Omega_0|\hat{\Phi}_0(x)\hat{\Phi}_0(y)\Omega_0\rangle \\
&= -i(2\pi)^{-3}\int dp\theta(p^0)\delta(p^2-m^2)e^{-ip(x-y)}
\end{aligned} \tag{3.30}$$

and

$$\Delta_-(x) \stackrel{\text{def}}{=} -\Delta_+(-x). \tag{3.31}$$

Since $\text{supp } \widetilde{\Delta}_\pm(p) \subset \overline{V}_\pm$, we may recursively define

$$\widetilde{\Delta}_\pm^{n+1}(p) \stackrel{\text{def}}{=} (2\pi)^{-2}\left(\widetilde{\Delta}_\pm^n * \widetilde{\Delta}_\pm\right)(p), \tag{3.32}$$

(see Exercise 49) in spite of the singularities²⁵ of $\Delta_\pm(x)$ on the light cone. Actually, this definition has to be used in Corollary 3.2.2 and to fix $\Delta_F^n(x)$ for $x \neq 0$:

$$\Delta_F^n(x) \stackrel{\text{def}}{=} \begin{cases} \Delta_+^n(x) & \text{for } x \in \mathbb{R}^4 \setminus \overline{V}_-, \\ (-1)^n\Delta_-^n(x) & \text{for } x \in \mathbb{R}^4 \setminus \overline{V}_+. \end{cases} \tag{3.33}$$

²⁴In (2.62) we wrote $\Delta_m^{(+)}$ instead of Δ_+ .

²⁵See (Bogoliubov and Shirkov, 1959, Sect. 15.2).

Supplemented by (3.33), (3.27) becomes a rigorous equation on $\{(x_1, \dots, x_n) \in \mathbb{R}^{4n} : x_\nu \neq x_\mu \text{ for } \nu \neq \mu\}$.

Exercise 49 Let $M = \overline{M} \subset \mathbb{R}^4$ and let $F, G \in \mathcal{S}(\mathbb{R}^4)'$ fulfill the conditions

$$\text{supp } \tilde{F} \subset \overline{V_+} \supset \text{supp } \tilde{G}$$

and

$$\text{supp } F \subset M.$$

Show that convolution of \tilde{F} with \tilde{G} is well-defined and that the Fourier transform FG of $(2\pi)^{-2} \tilde{F} * \tilde{G}$ fulfills the conditions

$$\text{supp } (FG) \subset M, \quad \text{supp } \widetilde{FG} \subset \overline{V_+}$$

and, if both F and G are sufficiently regular:

$$(FG)(x) = F(x)G(x) \quad \text{pointwise.}$$

The nontrivial problem is extension of (3.33) to all of \mathbb{R}^4 for $n > 1$. No doubt, (3.33) may be extended to a Lorentz invariant tempered distribution on all of \mathbb{R}^4 (see Sect. 3.2.2). This extension is unique up to addition of a Lorentz invariant distribution with point-like support at the origin. However, without further restrictions there is no hope to extract physically relevant information.

3.2.2 Counter Terms and Renormalization

The guiding heuristic principle for minimizing the arbitrariness in the definition of $\Delta_{\text{F}}(x)^n$ is to make it no more singular at the origin than necessary.²⁶ For $n = 1$ this means to take (3.28). For $n > 1$ the allowed (tempered) solutions may be constructed as follows:

One introduces a suitable covariant regularization $\Delta_{\text{F},M}$ of Δ_{F} depending on a parameter M such that for finite M the naive definition

$$\widetilde{\Delta_{\text{F},M}^n}(p) \stackrel{\text{def}}{=} (2\pi)^{-2(n-1)} \left(\widetilde{\Delta_{\text{F},M}} * \dots * \widetilde{\Delta_{\text{F},M}} \right) (p) \quad (3.34)$$

works and:

$$\begin{aligned} \int \Delta_{\text{F},M}(x) \varphi(x) dx &\xrightarrow{M \rightarrow \infty} \int \Delta_{\text{F}}(x) \varphi(x) dx \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^4), \\ \int \Delta_{\text{F},M}^n(x) \varphi(x) dx &\xrightarrow{M \rightarrow \infty} \begin{cases} \int \Delta_+^n(x) \varphi(x) dx & \forall \varphi \in \mathcal{S}(\mathbb{R}^4 \setminus \overline{V_-}) , \\ (-1)^n \int \Delta_-^n(x) \varphi(x) dx & \forall \varphi \in \mathcal{S}(\mathbb{R}^4 \setminus \overline{V_+}) . \end{cases} \end{aligned}$$

For given n this regularization has to fulfill the requirement that

$$\Delta_{\text{F},M}^n(x) = \sum_{\nu=1}^N A_{\nu,M} \square_x^{\nu-1} \delta(x)$$

²⁶See also (Epstein and Glaser, 1973, Sect. 5).

has a $(M \rightarrow \infty)$ -limit $\Delta_{\text{F,reg}}^n(x)$ in $\mathcal{S}(\mathbb{R}^4)'$ for a minimal number $N = N(n)$ of suitable sequences $A_{1,M}, \dots, A_{N,M}$. This way $\Delta_{\text{F,reg}}^n(x)$ is fixed up to addition of a distribution of the form

$$\sum_{\nu=1}^N c_\nu \square_x^{\nu-1} \delta(x)$$

with **finite** coefficients c_1, \dots, c_N . A suitable regularization is for example PAULI-VILLARS *regularization*

$$\begin{aligned} \Delta_{\text{F},M}(x) &\stackrel{\text{def}}{=} (2\pi)^{-4} \int \left(\frac{1}{p^2 - m^2 + i\epsilon} - \frac{1}{p^2 - M^2 + i\epsilon} \right) e^{-ipx} dp \\ &= (2\pi)^{-4} \int \frac{m^2 - M^2}{(p^2 - m^2 + i\epsilon)(p^2 - M^2 + i\epsilon)} e^{-ipx} dp. \end{aligned}$$

This gives²⁷

$$N(2) = 1, \quad N(3) = 2;$$

i.e.:

$$\begin{aligned} \Delta_{\text{F,reg}}^2(x) &= \lim_{M \rightarrow \infty} \left(\Delta_{\text{F},M}^2(x) - A_M \delta(x) \right), \\ \Delta_{\text{F,reg}}^3(x) &= \lim_{M \rightarrow \infty} \left(\Delta_{\text{F},M}^3(x) - B_M \delta(x) - C_M \square_x \delta(x) \right) \end{aligned}$$

in the topology of $\mathcal{S}(\mathbb{R}^4)'$ for suitable A_M, B_M, C_M .

Of course, these sequences have to diverge for $M \rightarrow \infty$ in order to compensate the so-called **ultraviolet divergences** appearing in (3.34) for $M \rightarrow \infty$. This is what is meant by the usual saying:

The ultraviolet infinities introduced by formal use of

$$\begin{aligned} \Delta_{\text{F}}^n(x) &= \Delta_{\text{F},\infty}^n(x) \\ &= (2\pi)^{-4(n+1)} \int \frac{1}{p_1^2 - m^2 + i\epsilon} \frac{1}{(p_2 - p_1)^2 - m^2 + i\epsilon} \cdots \\ &\quad \cdots \frac{1}{(p_n - p_{n-1})^2 - m^2 + i\epsilon} e^{-ip_n x} dp_n \cdots dp_1 \end{aligned}$$

can be removed by **infinite counter terms**, e.g. $A_\infty \delta(x)$ for $n = 2$ resp. $B_\infty \delta(x) + C_\infty \square_x \delta(x)$ for $n = 3$.

The essential result of the above considerations is the following:

For (3.26) in the limit²⁸ $g \rightarrow 1$ with $\hat{S}_1(x) = \lambda : \hat{\Phi}_0(x)^4 :$: any deviation of the working definition for Δ_{F}^n , used in (3.29), from the physically correct one (if such exists at all) may be compensated – at least up to second order in λ – by adding suitable **counter terms** C_ν of **higher order** in λ :

$$\begin{aligned} \lambda : \hat{\Phi}_0(x)^4 : &\longrightarrow \lambda : \hat{\Phi}_0(x)^4 : + C_1(\lambda) : \hat{\Phi}_0(x)^4 : + C_2(\lambda) : \hat{\Phi}_0(x)^2 : \\ &\quad + C_3(\lambda) : \hat{\Phi}_0(x) \square_x \hat{\Phi}_0(x) : + C_4(\lambda, x) \hat{1}. \end{aligned} \quad (3.35)$$

²⁷See (Bogoljubov and Šhirkov, 1984, Sects. 23.2 and 25.2).

²⁸We ignore the subtleties indicated by Exercise 52. See Section 3.3.2 in this connection.

The highly nontrivial²⁹ result of renormalization theory is that this compensation works for all orders of canonical perturbation theory for $\lambda: \hat{\Phi}_0(x)^4$:

Let us now indicate why such counter terms may be unavoidable in the construction of solutions to interacting field equations (see e.g. (Feldman and Raczka, 1977)).

We had already seen in Sect. 3.1.2 that the **formal** success of canonical quantization does not depend on the choice of the ‘physical’ mass m . It does not even depend on the normalization of the interacting field. More precisely, let $\hat{\Phi}(x)$ be a solution of

$$(\square + \check{m}_b) \hat{\Phi}(x) = -4\check{\lambda}:\hat{\Phi}(x)^3: \quad (3.36)$$

for $\check{m} = m_b$ and $\check{\lambda} = Z\lambda_b$ fulfilling the canonical commutation relations (3.16), the Hamiltonian being

$$\hat{H} = \frac{1}{2} \int_{x^0=0} : \left(\dot{\hat{\Phi}}(x)^2 + (\nabla \hat{\Phi}(x))^2 + m_b^2 \hat{\Phi}(x)^2 + 2Z\lambda_b \hat{\Phi}(x)^4 \right) : dx + \delta E \quad (3.37)$$

(compare Sect. 3.1.2). Then

$$\hat{\Phi}_Z(x) \stackrel{\text{def}}{=} \sqrt{Z} \hat{\Phi}(x)$$

fulfills (3.36) for $\check{m} = m_b$ and $\check{\lambda} = \lambda_b$. The Hamiltonian for $\hat{\Phi}_Z(x)$ is the same as for $\hat{\Phi}(x)$, of course, but in terms of $\hat{\Phi}_Z(x)$ it is given by

$$\hat{H} = \frac{1}{2Z} \int_{x^0=0} : \left(\dot{\hat{\Phi}}_Z(x)^2 + (\nabla \hat{\Phi}_Z(x))^2 + m_b^2 \hat{\Phi}_Z(x)^2 + 2\lambda_b \hat{\Phi}_Z(x)^4 \right) : dx + \text{const}. \quad (3.38)$$

Exercise 50 Show that (3.38) and

$$\hat{\Pi}_Z(x) \stackrel{\text{def}}{=} \frac{1}{Z} \dot{\hat{\Phi}}_Z(x) = \frac{1}{\sqrt{Z}} \dot{\hat{\Phi}}(x)$$

correspond to the canonical formalism for the (classical) Lagrangian

$$L(\Phi_Z, \dot{\Phi}_Z) = \frac{1}{2Z} \int_{x^0=0} \left(\dot{\Phi}_Z(x)^2 - (\nabla \Phi_Z(x))^2 - m_b^2 \Phi_Z(x)^2 - 2\lambda_b \Phi_Z(x)^4 \right) dx.$$

Therefore, given a solution $\hat{\Phi}_Z(x)$ of (3.36) for $\check{m} = m_b$ and $\check{\lambda} = \lambda_b$ with asymptotic ‘free’ time evolution governed by³⁰

$$\hat{H}_0 = \frac{1}{2} \int_{x^0=0} : \left(\dot{\hat{\Phi}}_0(x)^2 + (\nabla \hat{\Phi}_0(x))^2 + m^2 \hat{\Phi}_0(x)^2 \right) : dx$$

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²⁹A rough idea of how to proceed (in the BOGOLIUBOV-PARASIUK-HEPP sense) may be extracted from Sect. 3.3.2. Considerable complication is caused by so-called *overlapping divergences*.

³⁰In principle, of course, $\hat{\Phi}_Z(x)$ could be associated with several (asymptotic) particles of different masses.

(compare (3.21)), it may be necessary to scale $\hat{\Phi}_Z(x)$ to $\hat{\Phi}(x)$ in order to have (3.37) and

$$\hat{\Phi}(0, \mathbf{x}) = \hat{\Phi}_0(0, \mathbf{x}), \quad \dot{\hat{\Phi}}(0, \mathbf{x}) = \dot{\hat{\Phi}}_0(0, \mathbf{x}),$$

(at least up to local equivalence, if possible at all) where $\hat{\Phi}_0(\mathbf{x})$ denotes the free field with mass m (not m_b). Then in (3.23), **formally** assuming (3.13), we get

$$\begin{aligned} \hat{H}_1(t) &= \int_{x^0=t} : \left(-\frac{1}{2} \delta m^2 \hat{\Phi}_0(x)^2 + Z \lambda_b \hat{\Phi}_0(x)^4 \right) : d\mathbf{x} + \delta E \\ &= \int_{x^0=t} : \left(-\frac{1}{2} \delta m^2 \hat{\Phi}_0(x)^2 + \lambda \hat{\Phi}_0(x)^4 + \delta \lambda \hat{\Phi}_0(x)^4 \right) : d\mathbf{x} + \delta E, \end{aligned} \quad (3.39)$$

where

$$\delta m^2 \stackrel{\text{def}}{=} m^2 - m_b^2, \quad \delta \lambda \stackrel{\text{def}}{=} Z \lambda_b - \lambda,$$

λ denoting the *physical* (i.e. the *renormalized*) *coupling constant* to be fixed by some convention. Introduction of the counter terms³¹

$$-\frac{1}{2} \delta m^2 : \hat{\Phi}_0(x)^2 : , \quad \delta \lambda : \hat{\Phi}_0(x)^4 : , \quad \delta E$$

in (3.39) corresponds³² to (3.35). Correct choice of the counter terms is called **renormalization**.

In perturbation theory the coefficients $\delta m^2, \delta \lambda, \delta E$ are considered as power series in λ with coefficients depending on m . Their choice has to be adapted to the working definition of time ordering to meet physical requirements, especially stability of the vacuum and 1-particle states and, formally:³³

$$\delta m^2|_{\lambda=0} = \delta \lambda|_{\lambda=0} = \delta E|_{\lambda=0} = 0. \quad (3.40)$$

Exercise 51 For $s > 0$ show³⁴ that also

$$\hat{\Phi}(x) \stackrel{\text{def}}{=} s \hat{\Phi}_1(sx)$$

fulfills the canonical commutation relations (3.16) with $\hat{\Pi}(x) = \partial_0 \hat{\Phi}(x)$ and the field equation (3.36) for $\hat{m} = sm_b, \check{\lambda} = \lambda_b$. Moreover, show that the Hamiltonian for

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³¹Thanks to the counter terms there is now some chance that the r.h.s. of (3.37) may be rigorously defined by suitable limiting procedures as in lower-dimensional constructive field theory. However, the counter terms cannot all be finite, because of HAAG's theorem.

³²Recall that $:\hat{\Phi}_0(x) \square \hat{\Phi}_0(x): = -m^2 : \hat{\Phi}_0(x)^2 : .$ Therefore an arbitrary term proportional to $:\hat{\Phi}_0(x) \square \hat{\Phi}_0(x):$ can be extracted from $\delta m^2 : \hat{\Phi}_0(x)^2 :$ and compensated by change of δm^2 .

³³Actually, as pointed out for Equation (3.35), δm^2 and $\delta \lambda$ are assumed to be of higher order in λ .

³⁴This exercise indicates that – as far as perturbative calculations of S elements are concerned – taking a mass different from the physical mass m for the free field could be balanced by suitable change of the counter terms.

$\hat{\Phi}(x)$ is

$$\hat{H} = \frac{1}{2} \int_{x^0=0} : \left(\sum_{\mu=0}^3 \left(\partial_{\mu} \hat{\Phi}(x) \right)^2 + (sm_b)^2 \hat{\Phi}(x)^2 + 2\lambda_b \hat{\Phi}(x)^4 \right) : dx + \text{const.}$$

Exercise 52 Show that $\int : \hat{\Phi}_0(x)^4 : dx$ exists as a quadratic form but not as an operator on D_0 . Moreover, show that $\int : \hat{\Phi}_0(x)^2 : dx$ does not even exist as a quadratic form on D_0 .

3.2.3 FEYNMAN Rules

The perturbative expansion (3.24) for (3.39) – evaluated by Coroll. 3.2.2 – may be represented by diagrams. For these diagrams we need three types of vertices which we will draw as \bullet , \circ , and \odot . Each vertex of type \bullet or \odot is connected to exactly four solid lines, each vertex of type \circ to exactly two solid lines. These lines may either have free ends (*external lines*) or connect to another vertex (*internal lines*). Finally, the vertices of such a diagram G have to be indexed from 1 to V_G , where

$$V_G \stackrel{\text{def}}{=} \text{number of vertices of } G.$$

diagrams of this kind will be called *admitted* (for $\lambda\hat{\Phi}_4^4$ -perturbation theory in x -space).

To write down the formal operator \hat{A}_G represented by an admitted diagram G one has to apply the following FEYNMAN *rules*:

1. Write down a factor

$$\begin{aligned} & -i\lambda \quad \text{for each vertex } \bullet, \\ & \frac{i}{2}\delta m \quad \text{for each vertex } \circ, \\ & -i\delta\lambda \quad \text{for each vertex } \odot. \end{aligned}$$

2. For every pair of vertices with indices ν and μ , if directly connected by at least one internal line, write down a factor

$$i^{f_{\nu\mu}} \Delta_{\mathbb{F}}^{f_{\nu\mu}}(x_{\nu} - x_{\mu}),$$

where $f_{\nu\mu}$ is the number of internal lines directly connecting these vertices.

3. Multiply by the symmetry factor³⁵

$$\prod_{1 \leq \nu < \mu \leq V_G} \frac{1}{f_{\nu\mu}!} \prod_{1 \leq \alpha \leq V_G} \frac{l_{\alpha}!}{\left(l_{\alpha} - \sum_{\beta=1}^{V_G} f_{\alpha\beta} \right)!},$$

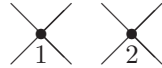
³⁵Recall the evaluation of Fig. 3.2.

where

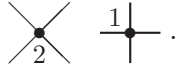
$$l_\alpha \stackrel{\text{def}}{=} \begin{cases} \text{number of lines} \\ \text{attached to vertex } \alpha. \end{cases}$$

4. For every external line write down a field operator $\hat{\Phi}_0(x_\nu)$, where ν is the index of the vertex to which the line is attached.
5. Normal order the resulting monomial and integrate all field variables over \mathbb{R}^4 .

Any two diagrams have to be considered as **equal** if they differ only by their diagrammatical realization.³⁶ For example,



is to be considered as equal to

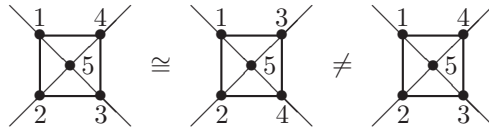


Then, for suitable³⁷ φ , we have³⁸

$$e^{i\varphi} \hat{S}_0 = \hat{1} + \sum_{G \text{ admitted}} \frac{1}{V_G!} \hat{A}_G \tag{3.41}$$

as an equation for formal power series in λ of quadratic forms on D_0 .

Any two diagrams G_1, G_2 are called **equivalent** ($G_1 \cong G_2$) if they differ only by the distribution of their vertex indices; e.g.:



Then the number of elements in the equivalence class $[G]$ of a diagram G is $\frac{V_G!}{I_G}$, where

$$I_G \stackrel{\text{def}}{=} \begin{cases} \text{number of permutations of the vertex indices} \\ \text{that do not change the diagram } G. \end{cases}$$

Therefore³⁹ (3.41) is equivalent to

$$e^{i\varphi} \hat{S}_0 = \hat{1} + \sum_{[G]} \frac{1}{I_G} \hat{A}_G. \tag{3.42}$$

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³⁶Consider the lines as elastic strings. Then all elastic deformations leave the diagrams unchanged.

³⁷Actually, everything should be defined with suitable cutoffs first. When removing these limits φ becomes infinite and compensates the infinite contributions of the **vacuum diagrams**, i.e. those diagrams G for which $\hat{A}_G \sim \hat{1}$. This will be used in (3.44).

³⁸Thanks to (3.40), only a finite number of diagrams contributes to each order in λ on the r.h.s. of (3.41).

³⁹Note that $G_1 \cong G_2 \implies \hat{A}_{G_1} = \hat{A}_{G_2}$, thanks to integration over all field variables, even though $G_1 \cong G_2$ does not imply $G_1 = G_2$, in general.

Denote by $G_1 \cdots G_N$ the diagram consisting of the disjoint subdiagrams G_1, \dots, G_N with natural renumbering of their vertices. Then

$$\hat{A}_{G_1 \cdots G_N} = : \hat{A}_{G_1} \cdots \hat{A}_{G_N} :$$

and

$$I_{G_1 \cdots G_N} = I_{G_1} \cdots I_{G_N} E_{G_1 \cdots G_N},$$

where

$$E_{G_1 \cdots G_N} \stackrel{\text{def}}{=} \begin{cases} \text{number of permutations } \pi \in S_N \text{ with} \\ ([G_{\pi 1}], \dots, [G_{\pi N}]) = ([G_1], \dots, [G_N]) . \end{cases}$$

Thus we have, formally,

$$\frac{1}{I_{G_1 \cdots G_N}} \hat{A}_{G_1 \cdots G_N} = \frac{1}{E_{G_1 \cdots G_N}} : \frac{1}{I_{G_1}} \hat{A}_{G_1} \cdots \frac{1}{I_{G_N}} \hat{A}_{G_N} : \quad (3.43)$$

for every set of diagrams $\{G_1, \dots, G_N\}$.

An admitted diagram is called **connected** if any two vertices are connected by a chain of internal lines. Obviously, for every diagram G there is a unique N -tuple of connected diagrams (G_1, \dots, G_N) with $G = G_1 \cdots G_N$. Consequently, by (3.43), (3.42) may be written as

$$e^{i\varphi} \hat{S}_0 = : \exp \left(\sum_{\substack{[G] \\ G \text{ connected}}} \frac{1}{I_G} \hat{A}_G \right) : .$$

With the physically natural requirement⁴⁰

$$\langle \Omega_0 | \hat{S}_0 \Omega_0 \rangle = 1$$

(*stability of the vacuum*) this implies:

$$\boxed{\begin{aligned} \hat{S}_0 &= : \exp \left(\sum_{[G] \in \mathcal{G}} \frac{1}{I_G} \hat{A}_G \right) : = : \exp \left(\sum_{\substack{G \text{ admitted} \\ [G] \in \mathcal{G}}} \frac{1}{V_G!} \hat{A}_G \right) : , \\ \text{where:} \\ \mathcal{G} &\stackrel{\text{def}}{=} \left\{ [G] : G \text{ connected, } \hat{A}_g \not\sim \hat{1} \right\} . \end{aligned}} \quad (3.44)$$

In order to evaluate (3.44) one has to fix, first of all, δm as a power series in λ depending on m and $\delta\lambda$. For this the physical requirement

$$\langle \Psi | \hat{S}_0 \Psi \rangle = \langle \Psi | \Psi \rangle \quad \forall \Psi \in \mathcal{H}_0^{(1)}$$

(*stability of 1-particle states*) is sufficient. Finally, one has to fix $\delta\lambda$ as a power series in λ depending only on m . For this some convention concerning the 2-2-scattering amplitude – depending on the preferred technical interpretation of the coupling constant – is necessary.

⁴⁰Actually, only $\langle \Omega_0 | \hat{S}_0 \Omega_0 \rangle = \hat{1}$ has to be required.

Exercise 53 Calculate the total cross section⁴¹

$$\sigma(p_1, p_2) = \frac{(2\pi)^2}{4\sqrt{(p_1 p_2)^2 - m^4}} \int_{q_j^0 = \omega_{\mathbf{q}_j}} \left| \frac{\langle \hat{a}_0^*(\mathbf{q}_1) \hat{a}_0^*(\mathbf{q}_2) \Omega_0 | (\hat{S}_0 - \hat{1}) \hat{a}_0^*(\mathbf{p}_1) \hat{a}_0^*(\mathbf{p}_2) \Omega_0 \rangle}{\delta(q_1 + q_2 - p_1 - p_2)} \right|^2 \times \delta(q_1 + q_2 - p_1 - p_2) \frac{d\mathbf{q}_1 d\mathbf{q}_2}{2q_1^0 2q_2^0}$$

of elastic scattering of two particles with initial 4-momenta p_1, p_2 to first order in the renormalized coupling constant λ .

Remark: The derivation of (3.44) also shows, that one may set $\varphi = 0$ in (3.41) if summation is restricted to those diagrams which do not have disjoint parts without any external line.

3.3 BOGOLIUBOV-SHIRKOV Theory

3.3.1 Basic Assumptions

BOGOLIUBOV and SHIRKOV (Bogoliubov and Shirkov, 1959) assume that there is a whole family of (interaction picture) S -matrices $\hat{S}_0(g)$ depending sufficiently smoothly on $g \in \mathcal{S}(\mathbb{R}^4, \mathbb{R})$, where $\mathcal{S}(\mathbb{R}^4, \mathbb{R})$ denotes the subspace of real-valued elements of $\mathcal{S}(\mathbb{R}^4)$. If

$$0 \leq g(x) \leq 1 \quad \forall x \in \mathbb{R}^4 \quad (3.45)$$

then $g(x)$ is interpreted as a degree to which the interaction is ‘switched on’ at x (formally by replacing the renormalized coupling constant λ in (3.39) by $g(x)\lambda$). $\hat{S}_0(g)$ is assumed to fulfill the following conditions for all $g \in \mathcal{S}(\mathbb{R}^4)$ fulfilling (3.45):

1. $\hat{S}_0(g)$ is unitary.
2. $\hat{S}_0(0) = \hat{1}$.
3. **Relativistic Covariance:**⁴²

$$\hat{U}_0(a, \Lambda) \hat{S}_0(g) \hat{U}_0(a, \Lambda)^{-1} = \hat{S}_0(\{a, \Lambda\}g) \quad \forall (a, \Lambda) \in \mathcal{P}_+^\uparrow,$$

where

$$(\{a, \Lambda\}g)(x) \stackrel{\text{def}}{=} g(\Lambda^{-1}(x - a)).$$

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⁴¹For a derivation of this formula see e.g. (Itzykson and Zuber, 1980a, Sect. 5-1-1). For its evaluation see also 4.3.2.

⁴²For theories with fermions a representation of $\text{iSL}(2, \mathbb{C})$ has to be used; see 4.2.2.

4. BOGOLIUBOV-SHIRKOV *causality*:

$$(x^0 < y^0 \forall (x, y) \in \text{supp } g_1 \times \text{supp } g_2) \implies \hat{S}_0(g_1 + g_2) = \hat{S}_0(g_2)\hat{S}_0(g_1).$$

Moreover, the functional derivatives

$$\hat{S}_n(x_1, \dots, x_n) \stackrel{\text{def}}{=} \left(\frac{\delta}{\delta g(x_1)} \cdots \frac{\delta}{\delta g(x_n)} \hat{S}_0(g) \right)_{|g=0}$$

are assumed to exist as operator-valued distributions on $\mathcal{S}(\mathbb{R}^{4n})$ with invariant dense domain D_0 (defined by (2.27)) for the ‘smeared’ $\hat{S}_n(x_1, \dots, x_n)$ and their adjoints. With this definition we have the following formal Taylor expansion.⁴³

$$\begin{aligned} \hat{S}_0(g) &= \left(\exp \left(\int dx g(x) \frac{\delta}{\delta \check{g}(x)} \right) \hat{S}_0(\check{g}) \right)_{|\check{g}=0} \\ &= \underbrace{\hat{1}}_{=\hat{S}_0(0)} + \sum_{n=1}^{\infty} \frac{1}{n!} \int \hat{S}_n(x_1, \dots, x_n) g(x_1) \cdots g(x_n) dx_1 \cdots dx_n. \end{aligned}$$

The $\hat{S}_n(x_1, \dots, x_n)$ are the central objects of the BOGOLIUBOV-SHIRKOV theory.⁴⁴ Its advantage is that it does not depend on the full-fledged interaction picture.⁴⁵

An immediate consequence of their definition is the permutation symmetry of the \hat{S}_n :

$$\boxed{\hat{S}_n(x_1, \dots, x_n) = \hat{S}_n(x_{\pi_1}, \dots, x_{\pi_n}) \forall \pi \in \mathcal{S}_n.} \quad (3.46)$$

Therefore

$$\hat{S}(M) \stackrel{\text{def}}{=} \begin{cases} \hat{S}_n(\xi_1, \dots, \xi_n) & \text{for } M = \{\xi_1, \dots, \xi_n\} \neq \emptyset \\ \hat{1} & \text{for } M = \emptyset \end{cases}$$

is a consistent definition (if the M are considered as sets of \mathbb{R}^4 -variables).

Exercise 54 Given the formal power series

$$\begin{aligned} S(g) &= S(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int S(\{x_1, \dots, x_n\}) g(x_1) \cdots g(x_n) dx_1 \cdots dx_n, \\ R(g) &= R(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int R(\{x_1, \dots, x_n\}) g(x_1) \cdots g(x_n) dx_1 \cdots dx_n \end{aligned}$$

in g , show that

$$S(g) R(g) = (S * R)(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int (S * R)(\{x_1, \dots, x_n\}) g(x_1) \cdots g(x_n) dx_1 \cdots dx_n$$

⁴³Many properties of the \hat{S}_n may be easily read off from this formal power series.

⁴⁴See (Stora, 1971; Epstein and Glaser, 1973) for a more elaborated version.

⁴⁵This does not mean, however, that it is physically better motivated.

holds (in the sense of formal power series) with the convolution product⁴⁶

$$(S * R)(X) \stackrel{\text{def}}{=} \sum_{M \subset X} S(M)R(X \setminus M) \quad \text{for } X \subset \{x_1, x_2, x_3, \dots\}.$$

Relativistic covariance implies **formal covariance**:

$$\boxed{\hat{U}_0(a, \Lambda) \hat{S}_n(x_1, \dots, x_n) \hat{U}_0(a, \Lambda)^{-1} = \hat{S}_n(\Lambda x_1 + a, \dots, \Lambda x_n + a)}. \quad (3.47)$$

Unitarity of $\hat{S}_0(g)$ implies that all functional derivatives of $\hat{V}(g) = \hat{S}_0(g) \hat{S}_0(g)^*$ vanish at g fulfilling (3.45). For $g = 0$ this gives

$$\begin{aligned} 0 &= \left(\frac{\delta}{\delta g(x_1)} \cdots \frac{\delta}{\delta g(x_n)} \left(\hat{S}_0(g) \hat{S}_0(g)^* \right) \right)_{|g=0} \\ &= \sum_{M \subset X_n} \underbrace{\left(\left(\prod_{x \in M} \frac{\delta}{\delta g(x)} \right) \hat{S}_0(g) \right)_{|g=0}}_{=\hat{1} \text{ for } M=\emptyset} \left(\left(\prod_{x' \in X_n \setminus M} \frac{\delta}{\delta g(x')} \right) \hat{S}_0(g)^* \right)_{|g=0}, \quad (3.48) \\ & \qquad \qquad \qquad X_n \stackrel{\text{def}}{=} \{x_1, \dots, x_n\}. \end{aligned}$$

Therefore, since

$$\left(\frac{\delta}{\delta g(x_1)} \cdots \frac{\delta}{\delta g(x_n)} \hat{S}(g)^* \right)_{|g=0} = \hat{S}_\nu(x_1, \dots, x_n)^*,$$

we have **formal unitarity**:⁴⁷

$$\boxed{0 = \sum_{M \subset X_n} \hat{S}(M) \hat{S}(X_n \setminus M)^* \quad \text{for } X_n = \{x_1, \dots, x_n\}, \quad n > 0.} \quad (3.49)$$

Let $g_1, g_2 \in \mathcal{S}(\mathbb{R}^4, \mathbb{R})$ fulfill the conditions

$$\begin{aligned} g_1(x), g_2(x) &\in [0, 1] \quad \forall x \in \mathbb{R}^4, \\ x^0 < y^0 &\quad \forall (x, y) \in \text{supp } g_1 \times \text{supp } g_2. \end{aligned} \quad (3.50)$$

Then BOGOLIUBOV-SHIRKOV causality and unitarity imply

$$\hat{S}_0(g_1 + g_2) \hat{S}_0(g_1)^* = \hat{S}_0(g_2) \quad (3.51)$$

and therefore

$$(3.50) \implies \frac{\delta}{\delta g_1(x)} \left(\frac{\delta \hat{S}_0(g)}{\delta g(y)} \hat{S}_0(g_1)^* \right)_{|g=g_1+g_2} = 0 \quad \text{for } (x, y) \in \text{supp } g_1 \times \text{supp } g_2. \quad (3.52)$$

⁴⁶This convolution product has many useful applications; see e.g. (Stora, 1971; Borchers, 1972; Doebner and Lücke, 1977; Hegerfeldt, 1985).

⁴⁷Since unitarity means $\hat{S}_0(g) \hat{S}_0(g)^* = \hat{1} = \hat{S}_0(g)^* \hat{S}_0(g)$, also $0 = \sum_{M \subset X_n} \hat{S}(M)^* \hat{S}(X_n \setminus M)$ holds.

Remark: Conversely, (3.52) implies

$$\partial_{\lambda_1} \partial_{\lambda_2} \left(\hat{S}_0(\lambda_1 g_1 + \lambda_2 g_2) \hat{S}_0(\lambda_1 g_1)^* \right) = 0 \quad \forall \lambda_1, \lambda_2 \in [0, 1]$$

hence constancy of

$$\partial_{\lambda_2} \left(\hat{S}_0(\lambda_1 g_1 + \lambda_2 g_2) \hat{S}_0(\lambda_1 g_1)^* \right)$$

in λ_1 and thus⁴⁸

$$\hat{S}_0(g_1 + g_2) \hat{S}_0(g_1)^* = \hat{S}_0(g_2) \underbrace{\hat{S}_0(0)^*}_{=1}$$

under the assumption (3.50). In other words: (3.52), thanks to unitarity and $\hat{S}_0(0) = \hat{1}$, implies BOGOLIUBOV-SHIRKOV causality.

Applying BOGOLIUBOV-SHIRKOV causality once more we get from (3.52)

$$(3.50) \implies \left(\frac{\delta}{\delta g(x)} \left(\frac{\delta \hat{S}_0(g)}{\delta g(y)} \hat{S}_0(g)^* \right) \right)_{|g=g_1+g_2} = 0 \text{ for } (x, y) \in \text{supp } g_1 \times \text{supp } g_2.$$

Evaluating this — or directly (3.52) — at $g = 0$ gives

$$\hat{K}(y; X_n) = 0 \quad \text{for } \max \{y^0 - x_1^0, \dots, y^0 - x_n^0\} > 0,$$

where

$$\hat{K}(y; X_n) \stackrel{\text{def}}{=} \frac{\delta}{\delta g(x_1)} \cdots \frac{\delta}{\delta g(x_n)} \left(\frac{\delta \hat{S}_0(g)}{\delta g(y)} \hat{S}_0(g)^* \right)_{|g=0},$$

i.e.

$$\hat{K}(y; X_n) = \sum_{M \subset X_n} \hat{S}(\{y\} \cup M) \hat{S}(X_n \setminus M)^*.$$

This way, by BOGOLIUBOV-SHIRKOV causality and relativistic covariance, we get **formal causality**:

$$\boxed{\sum_{M \subset X_n} \hat{S}(\{y\} \cup M) \hat{S}(X_n \setminus M)^* = 0 \quad \text{if } y - x \in \mathbb{R}^4 \setminus \overline{V_-} \quad \text{for some } x \in X_n, n > 0.} \quad (3.53)$$

Exercise 55 Show that (3.53) implies

$$\boxed{\frac{\delta}{\delta g(x)} \left(\frac{\delta \hat{S}_0(g)}{\delta g(y)} \hat{S}_0(g)^* \right) = 0 \text{ for } y - x \in \mathbb{R}^4 \setminus \overline{V_-}}$$

in the sense of formal power series in g .

⁴⁸Since then $\frac{d}{d\lambda} \left(\hat{S}_0(g_1 + \lambda g_2) \hat{S}_0(g_1)^* - \hat{S}_0(\lambda g_2) \hat{S}_0(0)^* \right) = 0$.

3.3.2 General Solution

For $y = x_{n+1}$ (3.53) implies

$$\hat{S}(X_{n+1}) = - \sum_{\substack{M \subset X_n \\ M \neq X_n}} \hat{S}(\{x_{n+1}\} \cup M) \hat{S}(X_n \setminus M)^* \quad (3.54)$$

if $x_{n+1} - x \in \mathbb{R}^4 \setminus \overline{V_-}$ for some $x \in X_n$.

This shows that the $\hat{S}_n(x_1, \dots, x_n)$ are fixed on

$$\mathbb{R}_{\neq}^{4n} \stackrel{\text{def}}{=} \{(x_1, \dots, x_n) \in \mathbb{R}^{4n} : x_\nu \neq x_\mu \text{ for } \nu \neq \mu\} \quad (3.55)$$

for $n = 2, 3, \dots$ once

$$\boxed{\hat{L}_I(x) \stackrel{\text{def}}{=} -i\hat{S}_1(x)} \quad (3.56)$$

is given. Of course, $\hat{L}_I(x)$ has to meet certain requirements. (3.49) for $n = 1$ means

$$\boxed{\hat{L}_I(x) = \hat{L}_I(x)^*}. \quad (3.57)$$

Therefore evaluation of (3.54) for $n = 1$ gives

$$\hat{S}(\{x_1, x_2\}) = \hat{S}_1(x_2)\hat{S}_1(x_1) = -\hat{L}_I(x_2)\hat{L}_I(x_1) \quad \text{for } x_2 - x_1 \in \mathbb{R}^4 \setminus \overline{V_-}. \quad (3.58)$$

Since this does not depend on the choice of indices, we conclude that⁴⁹

$$\boxed{[\hat{L}_I(x), \hat{L}_I(y)]_- = 0 \quad \text{for } x \times y}. \quad (3.59)$$

By (3.47), finally, we have

$$\boxed{\hat{U}(a, \Lambda)\hat{L}_I(x)\hat{U}(a, \Lambda)^{-1} = \hat{L}_I(\Lambda x + a)}. \quad (3.60)$$

Summing up:

$\hat{L}_I(x)$ must be a hermitian, scalar, local operator field.

Conversely, these properties of $\hat{L}_I(x)$ guarantee that (3.46), (3.47), (3.49), and (3.53) are fulfilled on \mathbb{R}_{\neq}^{4n} by⁵⁰

$$\hat{S}_n^T(x_1, \dots, x_\nu) \stackrel{\text{def}}{=} T \left(\hat{S}_1(x_1) \cdots \hat{S}_1(x_\nu) \right) \quad \text{for } \nu = 2, 3, \dots \quad (3.61)$$

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⁴⁹Note that, $x \times y \implies x_2 - x_1 \in \mathbb{R}^4 \setminus \overline{V_-} \ni x_1 - x_2$.

⁵⁰(3.49), for example, is obvious for $n = 1$ and therefore follows for $n = 2, 3, \dots$ – if restricted to \mathbb{R}_{\neq}^{4n} – because

$$\begin{aligned} & \sum_{M' \subset X_{n+1}} \hat{S}^T(M') \hat{S}^T(X_{n+1} \setminus M')^* \\ &= \hat{S}_1^T(x_{n+1}) \sum_{M \subset X_n} \hat{S}^T(M) \hat{S}^T(X_n \setminus M)^* + \sum_{M \subset X_n} \hat{S}^T(M) \hat{S}^T(X_n \setminus M)^* \hat{S}_1^T(x_{n+1})^* \end{aligned}$$

holds on the subregion of $\mathbb{R}_{\neq}^{4(n+1)}$ characterized by: $x_{n+1}^0 > x_j^0$ for $j = 1, \dots, n$. For (3.53) see also (Bogoliubov and Shirkov, 1959, Sect. 18.5).

if T is a (linear) covariant time ordering operation, i.e. fulfills the conditions

$$T\left(\hat{S}_1(x_1)\cdots\hat{S}_1(x_\nu)\right) = \hat{S}_1(x_{\pi_1})\cdots\hat{S}_1(x_{\pi_\nu}) \quad \text{for } \pi \in S_\nu \text{ with } x_{\pi_1}^0 > \dots > x_{\pi_\nu}^0 \quad (3.62)$$

and

$$\hat{U}_0(a, A)T\left(\hat{S}_1(x_1)\cdots\hat{S}_1(x_\nu)\right)\hat{U}_0(a, A)^{-1} = T\left(\hat{S}_1(\Lambda x_1 + a)\cdots\hat{S}_1(\Lambda x_\nu + a)\right). \quad (3.63)$$

(on all of $\mathbb{R}^{4\nu}$ for $\nu = 2, 3, \dots$). Therefore:⁵¹

On the restricted region \mathbb{R}_{\neq}^{4n} there is no other choice for $\hat{S}_n(x_1, \dots, x_n)$ than (3.61).

The difficult problem is physically correct extension of $\hat{S}_n(x_1, \dots, x_n)$ to all of \mathbb{R}^{4n} ($n = 2, 3, \dots$).

If the $\hat{S}_\nu(x_1, \dots, x_\nu)$ are known for $\nu \leq n$ then $\hat{S}_{n+1}(x_1, \dots, x_{n+1})$ is fixed by (3.54) even on the complement of

$$\mathbb{R}_{=}^{4(n+1)} \stackrel{\text{def}}{=} \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{4(n+1)} : x_1 = x_2 = \dots = x_{n+1}\},$$

since there must be a pair $x_\nu, x_\mu \in \{x_1, \dots, x_{n+1}\}$ for which $x_\nu - x_\mu \in \mathbb{R}^4 \setminus \overline{V_-}$ unless $x_1 = x_2 = \dots = x_n$. In other words:

If $\hat{S}_1(x), \dots, \hat{S}_n(x_1, \dots, x_n)$ are fixed then $\hat{S}_{n+1}(x_1, \dots, x_{n+1})$ is unique up to addition of an operator field $\hat{A}_{n+1}(x_1, \dots, x_{n+1})$ which is *quasilocal*, i.e.

$$\text{supp } \hat{A}_{n+1}(x_1, \dots, x_{n+1}) \subset \mathbb{R}_{=}^{4(n+1)}.$$

Further details: Similarly to (3.52), starting from

$$\hat{S}_0(g_2)^* \hat{S}_0(g_1 + g_2) = \hat{S}_0(g_1)$$

instead of (3.51), we get

$$(3.50) \implies \frac{\delta}{\delta g_1(x)} \left(\hat{S}_0(g_2)^* \frac{\delta \hat{S}_0(g)}{\delta g(x)} \right)_{|g=g_1+g_2} = 0 \quad \text{for } (x, y) \in \text{supp } g_1 \times \text{supp } g_2$$

and hence

$$\sum_{M \subset X_n} \hat{S}(X_n \setminus M)^* \hat{S}(\{y\} \cup M) = 0 \quad \text{if } y - x \in \mathbb{R}^4 \setminus \overline{V_-} \text{ for some } x \in X_n, n > 0.$$

Therefore, the difference

$$D(X_n; y) \stackrel{\text{def}}{=} R(X_n; y) - A(X_n; y)$$

⁵¹Recall that (3.53) implies (3.54).

of the *retarded* function

$$R(X_n; y) \stackrel{\text{def}}{=} \sum_{M \subset X_n} \hat{S}(X_n \setminus M)^* \hat{S}(\{y\} \cup M)$$

and the *advanced* function

$$A(X_n; y) \stackrel{\text{def}}{=} \sum_{M \subset X_n} \hat{S}(\{y\} \cup M) \hat{S}(X_n \setminus M)^*$$

vanishes whenever one of the arguments x_ν is spacelike w.r.t. y . If all the $S(X_\nu)$ are known for all $\nu \leq n$ then also $D(X_n; x_{n+1})$ and $\hat{S}(X_{n+1}) - A(X_n; x_{n+1})$ are known and determination of $\hat{S}(X_{n+1})$ is equivalent to physically correct splitting of $D(X_n; x_{n+1})$ into an advanced function $A(X_n; x_{n+1})$ and a retarded function $R(X_n; x_{n+1})$. The main difficulty of such a splitting is to get $A(X_n; x_{n+1})$ and $R(X_n; x_{n+1})$ Lorentz covariant (see (Steinmann, 1963; Epstein, 1966)).

Note that one may work with $\hat{S}_0(x)^{-1}$ instead of $\hat{S}_0(x)^*$ and use (3.57) instead of formal unitarity (see (Epstein and Glaser, 1973, Sections 1.2 and 2)).

Let us use this to analyze a given formal power series $\hat{S}_0(g)$ that fulfills all the BOGOLIUBOV-SHIRKOV requirements.

Let $\hat{A}_2, \dots, \hat{A}_n$ be quasilocal operators and let T be a (linear) time ordering operation, defined for all direct products \hat{C}_j composed of elements from $\{\hat{S}_1, \hat{A}_2, \dots, \hat{A}_n\}$, fulfilling the following requirements:

$$T\left(\hat{C}_1(x_1, \dots, x_j) \hat{C}_2(y_1, \dots, y_k)\right) = T\left(\hat{C}_1(x_1, \dots, x_j)\right) T\left(\hat{C}_2(y_1, \dots, y_k)\right) \quad (3.64)$$

if $\sup\{x_1^0, \dots, x_j^0\} > \sup\{y_1^0, \dots, y_k^0\}$,

$$\begin{aligned} T\left(\hat{C}_1(x_1, \dots, x_j) T\left(\hat{C}_2(y_1, \dots, y_k)\right)\right) &= T\left(T\left(\hat{C}_1(x_1, \dots, x_j)\right) \hat{C}_2(y_1, \dots, y_k)\right) \\ &= T\left(\hat{C}_1(x_1, \dots, x_j) \hat{C}_2(y_1, \dots, y_k)\right) \\ &= T\left(\hat{C}_2(y_1, \dots, y_k) \hat{C}_1(x_1, \dots, x_j)\right), \end{aligned} \quad (3.65)$$

$$\hat{U}_0(a, A) T\left(\hat{C}(x_1, \dots, x_\nu)\right) \hat{U}_0(a, A)^{-1} = T\left(\hat{C}(\Lambda x_1 + a, \dots, \Lambda x_\nu + a)\right). \quad (3.66)$$

Exercise 56 Show the following:

1. The \hat{A}_ν are local relative to \hat{S}_1 in the sense that

$$x \times y_1 \implies \left[\hat{S}_1(x), \hat{A}_\nu(y_{\pi 1}, \dots, y_{\pi \nu}) \right]_- = 0 \quad \forall \pi \in S_\nu.$$

2. The \hat{A}_ν are local relative w.r.t. each other in the sense that

$$x_1 \times y_1 \implies \left[\hat{A}_\mu(x_1, \dots, x_\mu), \hat{A}_\nu(y_{\pi 1}, \dots, y_{\pi \nu}) \right]_- = 0 \quad \forall \pi \in S_\nu.$$

3. The \hat{A}_ν transform according to

$$\hat{U}(a, \Lambda) \hat{A}_\nu(x_1, \dots, x_\nu) \hat{U}(a, \Lambda)^{-1} = \hat{A}_\nu(\Lambda x_1 + a, \dots, \Lambda x_\nu + a) \quad \forall (a, \Lambda) \in \mathcal{P}_+^\uparrow.$$

Then we define

$$\hat{S}_{\hat{A}_2, \dots, \hat{A}_n}^T(\{x_1, \dots, x_\nu\}) \stackrel{\text{def}}{=} \frac{\delta}{\delta g(x_1)} \cdots \frac{\delta}{\delta g(x_\nu)} \hat{S}_{\hat{A}_2, \dots, \hat{A}_n}^T(g)|_{g=0},$$

where

$$\begin{aligned} & \hat{S}_{\hat{A}_2, \dots, \hat{A}_n}^T(g) \\ & \stackrel{\text{def}}{=} T \left(\exp \left(\int \hat{S}_1(x) g(x) dx + \sum_{\nu=2}^n \int \hat{A}_\nu(x_1, \dots, x_\nu) g(x_1) \cdots g(x_\nu) dx_1 \cdots dx_\nu \right) \right) \end{aligned}$$

(in the sense of formal power series).

Exercise 57 Show that $\hat{S}_{\hat{A}_2, \dots, \hat{A}_n}^T(g)$ fulfills all the BOGOLIUBOV-SHIRKOV requirements if also

$$\hat{A}_\nu(x_{\pi_1}, \dots, x_{\pi_\nu}) = \hat{A}_\nu(x_1, \dots, x_\nu) = -\hat{A}_\nu(x_1, \dots, x_\nu)^* \quad \forall \pi \in S_\nu, \nu \leq n \quad (3.67)$$

holds.

Now assume

$$\hat{S}(\{x_1, \dots, x_\nu\}) = \hat{S}_{\hat{A}_2, \dots, \hat{A}_n}^T(\{x_1, \dots, x_\nu\}) \quad \text{for } \nu \leq n. \quad (3.68)$$

Then, according to the above considerations,

$$\hat{A}_{n+1}(x_1, \dots, x_{n+1}) \stackrel{\text{def}}{=} \hat{S}(\{x_1, \dots, x_{n+1}\}) - \hat{S}_{\hat{A}_2, \dots, \hat{A}_n}^T(\{x_1, \dots, x_{n+1}\}) \quad (3.69)$$

must be a quasilocal operator field, since (3.54) holds for both \hat{S} and $\hat{S}_{\hat{A}_2, \dots, \hat{A}_n}^T$.

Exercise 58 Show that, with \hat{A}_{n+1} defined by (3.69), the statements of Exercise 56 and (3.67) hold also for n replaced by $n + 1$.

If we can extend⁵² the T -operation to all direct products \hat{C}_j composed of elements from $\{\hat{S}_1, \hat{A}_2, \dots, \hat{A}_{n+1}\}$ in a way respecting (3.64)–(3.66) we get (3.68) for n replaced by $n + 1$. This way we are lead to the following BOGOLIUBOV-SHIRKOV

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⁵²In general, it is not at all obvious that such an extension is possible. Up to now, nobody could provide a proof for existence of the **covariant** T -products for interacting WIGHTMAN fields (compare (Steinmann, 1963; Epstein, 1966) to see the difficulties).

conjecture:

Let $\hat{S}_0(g)$ be a formal power series fulfilling all the BOGOLIUBOV-SHIRKOV requirements and let T be a (linear) covariant time ordering operation fulfilling (3.64)–(3.66) for arbitrary multiple direct products \hat{C}_j of \hat{S}_1 with itself. Then there is a sequence of quasilocal operators $\hat{A}_2, \hat{A}_3, \dots$ and a suitable extension of T for which

$$\hat{S}_0(g) = T\left(\exp\left(\int dx \hat{S}_1(x)g(x) + \hat{A}(g)\right)\right), \quad \text{where}$$

$$\hat{A}(g) \stackrel{\text{def}}{=} \sum_{\nu=2}^{\infty} \int \hat{A}_\nu(x_1, \dots, x_\nu) g(x_1) \cdots g(x_\nu) dx_1 \cdots dx_\nu,$$

holds in the sense of formal power series in g .

The transition

$$\int \hat{S}_1(x)g(x) dx \longrightarrow \int \hat{S}_1(x)g(x) dx + \hat{A}(g)$$

is to be considered as **renormalization** from the BOGOLIUBOV-SHIRKOV point of view. Without correct choice of the **counter terms** $\hat{A}(g)$ transition to the **adiabatic limit** ($g(x) \longrightarrow 1$) will typically not be possible (recall Exercise 52).

3.3.3 Generalization to Nonlocalizable Test Spaces

As already pointed out in 2.2.2, there is no physical justification for the technical requirement that $\hat{\Phi}_0(g)$ be defined for all **tempered** g . Therefore other test spaces should be taken under consideration. Convenient families of test spaces, parameterized by $s \geq 0$, are the following:⁵³

$$S^s(\mathbb{R}^{4n}) \stackrel{\text{def}}{=} \bigcup_{A>0} S^{s,A} \quad (\text{inductive limit}),$$

$$J^s(\mathbb{R}^{4n}) \stackrel{\text{def}}{=} \bigcap_{A>0} S^{s,A} \quad (\text{projective limit}),$$

where

$$S^{s,A}(\mathbb{R}^{4n}) \stackrel{\text{def}}{=} \left\{ \varphi \in \mathcal{S}(\mathbb{R}^{4n}) : \|\varphi\|_{A,N}^s < \infty \forall N \in \mathbb{N} \right\},$$

the topology of $S^{s,A}(\mathbb{R}^{4n})$ being given by the family of norms⁵⁴

$$\|\varphi\|_{A,N}^s \stackrel{\text{def}}{=} \sup_{\tilde{x} \in \mathbb{R}^{4n}} (1 + \|\tilde{x}\|)^N \sup_{\tilde{\alpha} \in \mathbb{Z}_+^{4n}} \left(A + \frac{1}{N}\right)^{-|\tilde{\alpha}|} \tilde{\alpha}^{-s\tilde{\alpha}} |\varphi^{(\tilde{\alpha})}(\tilde{x})|.$$

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⁵³EFIMOV (see (Efimov, 1968) and references given there) used slightly different test spaces.

⁵⁴We write

$$\tilde{x} = (x_1, \dots, x_n), \quad \|\tilde{x}\| = \sqrt{\sum_{\nu=1}^n \sum_{\mu=0}^3 (x_\nu^\mu)^2}, \quad \tilde{p}\tilde{x} = \sum_{\nu=1}^n \sum_{\mu=0}^3 p_\nu^\mu x_\nu^\mu$$

Note that

$$s_1 < s_2 \implies S^{s_1} \preccurlyeq S^{s_2},$$

where $S^{s_1} \preccurlyeq S^{s_2}$ means that S^{s_1} is contained in S^{s_2} as a set and that the topology of S^{s_1} is finer than that induced by S^{s_2} .

The elements of $S^s(\mathbb{R}^{4n})$ resp. $J^s(\mathbb{R}^{4n})$ may be characterized by their Fourier transforms

$$\tilde{\varphi}(\check{p}) \stackrel{\text{def}}{=} (2\pi)^{-2n} \int \varphi(\check{x}) e^{+i\check{p}\check{x}} d\check{x}$$

as follows:

$$\begin{aligned} \varphi \in S^s(\mathbb{R}^{4n}) &\iff \left(\exists A > 0 : \|\tilde{\varphi}\|_s^{A,N} < \infty \forall N \in \mathbb{N} \right), \\ \varphi \in J^s(\mathbb{R}^{4n}) &\iff \left(\forall A > 0 : \|\tilde{\varphi}\|_s^{A,N} < \infty \forall N \in \mathbb{N} \right). \end{aligned}$$

Here

$$\|\tilde{\varphi}\|_s^{A,N} \stackrel{\text{def}}{=} \sup_{\check{p} \in \mathbb{R}^{4n}} \sup_{\substack{\check{\alpha} \in \mathbb{Z}_+^{4n} \\ |\check{\alpha}| \leq N}} g_s \left(\left(A + \frac{1}{N} \right)^{-1} \|\check{p}\| \right) |\tilde{\varphi}^{(\check{\alpha})}(\check{p})|,$$

where

$$g_s(t) \stackrel{\text{def}}{=} \sup_{\mu \in \mathbb{Z}_+} \mu^{-s\mu} |t|^\mu \quad \text{for } t \in \mathbb{R}^1.$$

Note that

$$e^{-\frac{se}{2}} e^{\frac{s}{e}|t|^{1/s}} \leq g_s(|t|) \leq e^{\frac{s}{e}|t|^{1/s}} \quad \forall t \in \mathbb{R}^1,$$

if $s > 0$, while

$$g_0(t) = \begin{cases} 1 & \text{for } |t| \leq 1. \\ \infty & \text{else.} \end{cases}$$

This implies that $S^s(\mathbb{R}^{4n})$ contains only entire analytic functions if $s < 0$ and that

$$\varphi \in S^0(\mathbb{R}^{4n}) \iff \tilde{\varphi} \in \mathcal{D}(\mathbb{R}^{4n}) \quad , \quad J^0(\mathbb{R}^4) = \emptyset.$$

Moreover, the elements φ of $S^1(\mathbb{R}^{4n})$ are seen to be analytic in a complex neighborhood of \mathbb{R}^{4n} (depending on φ). Finally, as shown by Roumieu ([Roumieu, 1960](#); [Roumieu, 1963](#)) (see also ([Gelfand and Schilow, 1962](#), Kapitel IV)):

$$s > 1 \implies S^s(\mathbb{R}^{4n}) \cap \mathcal{D}(\mathbb{R}^{4n}) \text{ dense in } S^s(\mathbb{R}^{4n}).$$

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and use standard multi-index notation:

$$\begin{aligned} |\check{\alpha}| &= \sum_{\nu=1}^n \sum_{\mu=0}^3 \alpha_\nu^\mu, \quad \check{\alpha}^{\check{\beta}} = \prod_{\nu=1}^n \prod_{\mu=0}^3 (\alpha_\nu^\mu)^{\beta_\nu^\mu}, \quad \check{\alpha}^{-s\check{\beta}} = (\check{\alpha}^{\check{\beta}})^{-s}, \\ \varphi^{(\check{\alpha})}(\check{x}) &= D_{\check{x}}^{\check{\alpha}} \varphi(\check{x}), \quad D_{\check{x}}^{\check{\alpha}} = \prod_{\nu=1}^n \prod_{\mu=0}^3 \left(\frac{\partial}{\partial x_\nu^\mu} \right)^{\alpha_\nu^\mu}. \end{aligned}$$

Therefore the standard definition of *support* may be applied to generalized functions on $S^s(\mathbb{R}^{4n})$ if and only if $s > 1$. For the spaces $J^s(\mathbb{R}^{4n})$ the situation is quite similar.

For $s < 1$, if one wants to test by functions $\varphi \in S^s(\mathbb{R}^{4n})$ whether a given $F \in S^s(\mathbb{R}^{4n})'$ is ‘localized’ within a closed subset M of \mathbb{R}^{4n} , there is essentially only one possibility:

Check whether

$$F(\varphi_\nu) \xrightarrow{\nu \rightarrow \infty} 0$$

holds for every sequence $\varphi_1, \varphi_2, \dots$ with

$$\varphi_\nu \xrightarrow[\nu \rightarrow \infty]{\text{suitably}} 0 \quad \text{on } M.$$

The question is just how to specify what ‘suitably’ should mean, here.

Let us interpret ‘suitably’ as ‘ S^s -like’ in the sense that all the φ_ν are elements of $S^s(\mathbb{R}^{4n})$ and⁵⁵

$$\left(\varphi_\nu \xrightarrow[\nu \rightarrow \infty]{S^s\text{-like}} 0 \quad \text{on } M \right) \stackrel{\text{def}}{\iff} \left(\exists A > 0 : \|\varphi_\nu\|_{A,N}^{s,M} \xrightarrow{\nu \rightarrow \infty} 0 \quad \forall N \in \mathbb{N} \right),$$

where⁵⁶

$$\|\varphi\|_{A,N}^{s,M} \stackrel{\text{def}}{=} \sup_{\tilde{x} \in M} (1 + \|\tilde{x}\|)^N \sup_{\tilde{\alpha} \in \mathbb{Z}_+^{4n}} \left(A + \frac{1}{N} \right)^{-|\tilde{\alpha}|} \tilde{\alpha}^{-s\tilde{\alpha}} |\varphi^{(\tilde{\alpha})}(\tilde{x})|.$$

Let $F \in S^s(\mathbb{R}^{4n})'$ and $M = \overline{M} \subset \mathbb{R}^{4n}$. Then M is called a **quasi-support** of F with respect to $S^s(\mathbb{R}^{4n})$ if and only if

$$\left(\varphi_\nu \xrightarrow[\nu \rightarrow \infty]{S^s\text{-like}} 0 \quad \text{on } M \right) \implies F(\varphi_\nu) \xrightarrow{\nu \rightarrow \infty} 0$$

holds for arbitrary $\varphi_\nu \in S^s(\mathbb{R}^{4n})$. A neutral scalar quantum field $\hat{\Phi}(x)$ with domain D is called **essentially local** with respect to $S^s(\mathbb{R}^{4n})$ if and only if $\{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : x - y \in \overline{V}\}$ is a quasi-support of $\langle \Psi_1 | [\hat{\Phi}(x), \hat{\Phi}(y)]_- \Psi_2 \rangle$ with respect to $S^s(\mathbb{R}^{4n})$ for all $\Psi_1, \Psi_2 \in D$.

For $s > 1$ every quasi-support w.r.t. S^s contains the (ordinary) support as a subset, whereas for $s < 1$:

$$\left. \begin{array}{l} M_1 \text{ quasi-support of } F \\ \exists \epsilon > 0 : M_1 \subset U_\epsilon(M_2) \end{array} \right\} \implies M_2 \text{ quasi-support of } F.$$

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⁵⁵EFIMOV required the $\varphi_1, \varphi_2, \dots$ to be what he called *projecting sequences with support M* (see (Alebastrov and Efimov, 1974) and references given there). This especially means that the analytic continuations $\varphi_\nu(\tilde{z})$ of the $\varphi_\nu(\tilde{x})$ have to converge uniformly to zero in the region $M + i\mathbb{R}^{4n}$ and uniformly to one in every region of the form $K + i\mathbb{R}^{4n}$ with K a compact subset of $\mathbb{R}^{4n} \setminus M$.

⁵⁶Note that $\|\varphi\|_{A,N}^{s,\mathbb{R}^{4n}} = \|\varphi\|_{A,N}^s$, so that every element of $S^s(\mathbb{R}^{4n})'$ is localized in \mathbb{R}^{4n} , at least.

This is why we speak about ‘quasi’-supports. For instance, $\{0\}$ is a quasi-support of $F(x) = e^{-\|x\|^2}$ with respect to $S^{1/4}(\mathbb{R}^4)$. Thus, obviously, a quasi-support is not a domain of **strict** localization, in general. Nevertheless the *PCT* theorem and the spin-statistics theorem could be proved for fields which are only essentially local⁵⁷ with respect to $S^0(\mathbb{R}^4)$ (see (Lücke, 1984; Lücke, 1986)).

Now it is evident how to generalize the BOGOLIUBOV-SHIRKOV theory:

Formulate everything – except BOGOLIUBOV-SHIRKOV causality – with S^s instead of \mathcal{S} and replace BOGOLIUBOV-SHIRKOV causality by the requirement of **generalized** BOGOLIUBOV-SHIRKOV *causality*.⁵⁸

$$\boxed{\begin{array}{l} \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : x - y \in \overline{V_+}\} \text{ is a quasi-support} \\ \text{of } \frac{\delta}{\delta g(x)} \left(\frac{\delta \hat{S}_0(g)}{\delta g(y)} \hat{S}_0(g)^* \right)_{|g=0} \text{ with respect to } S^s(\mathbb{R}^8). \end{array}}$$

Naturally, we call $M = \overline{M}$ a quasi-support of $F \in J^s(\mathbb{R}^{4n})'$ with respect to $J^s(\mathbb{R}^{4n})$ if and only if

$$\left(\varphi_\nu \xrightarrow[\nu \rightarrow \infty]{J^s\text{-like}} 0 \quad \text{on } M \right) \implies F(\varphi_\nu) \xrightarrow[\nu \rightarrow \infty]{} 0,$$

where

$$\left(\varphi_\nu \xrightarrow[\nu \rightarrow \infty]{J^s\text{-like}} 0 \quad \text{on } M \right) \stackrel{\text{def}}{\iff} \left(\forall A, N > 0 : \|\varphi_\nu\|_{A,N}^{s,M} \xrightarrow[\nu \rightarrow \infty]{} 0 \right).$$

Now, one would like to have a convenient criterion for $\overline{V_+}$ being a quasi-support. For tempered distributions we have the following.

Theorem 3.3.1 *Let $\mathcal{L}(p+iq)$ be an analytic function on the tube $\mathbb{R}^4 + iV_+$ having the following property:*

For every $\eta \in V_+$ there is a polynomial P_η for fulfilling

$$|\mathcal{L}(p + i(\eta + a))| \leq |P_\eta(p + ia)| \quad \forall p \in \mathbb{R}^4, a \in V_+.$$

Then there is a tempered distribution $F(x)$ with

$$F(\varphi) = \lim_{\epsilon \rightarrow +0} \int \mathcal{L}(p + i(\epsilon, 0, 0, 0)) \tilde{\varphi}(-p) dp \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^4).$$

For this distribution

$$\text{supp } F \subset \overline{V_+}.$$

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⁵⁷In spite of the – generally misinterpreted – result of (Borchers and Pohlmeier, 1968) there are examples of **nonlocal** tempered fields which are essentially local w.r.t. $S^0(\mathbb{R}^4)$, as shown in (Bümmerstedt and Lücke, 1975, Sect. 5).

⁵⁸We choose the spaces J^s since we are primarily interested in the case $s = 1$, here. For the general study of axiomatic field theory the case $s = 0$ was the real challenge.

Conversely, for every tempered distribution with such support there is a unique function \mathcal{L} of the type specified above, namely the Laplace transform

$$\mathcal{L}(p + iq) = (2\pi)^{-2} \int F(x) e^{i(p+iq)x} dx.$$

Proof: See (Streater and Wightman, 1989, Theorem 2.8). ■

An immediate consequence of Theorem 3.3.1 is

$$\text{supp } \Delta_{\text{ret}}(x) \subset \overline{V_+}$$

for the *retarded commutator*

$$\Delta_{\text{ret}}(x) \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow +0} (2\pi)^{-4} \int \frac{e^{-ipx}}{(p^0 + i\epsilon)^2 - \mathbf{p}^2 - m^2} dp.$$

Note that, by Cauchy's integral theorem,

$$\Delta_{\text{F}}(x) = \Delta_{\text{ret}}(x) - \Delta_{-}(x)$$

for $\Delta_{\text{F}}(x)$ as defined in (3.28) and $\Delta_{-}(x)$ as defined in (3.30)/(3.31).

Exercise 59 Let $\{\mu_\alpha\}_{\alpha \in \mathbb{Z}_+^4}$ be a family of complex-valued Borel measures μ_α on \mathbb{R}^4 and assume

$$\sup_{\alpha \in \mathbb{Z}_+^4} A^{-|\alpha|} \alpha^{+s\alpha} \int (1 + \|x\|)^{-A} |\mu_\alpha|(dx) < \infty \quad \text{for } A > 0 \text{ large enough,}$$

where $|\mu|$ denotes the *variation* of the complex-valued measure μ :

$$|\mu|(E) \stackrel{\text{def}}{=} \sup \left\{ \left| \int_E f d\mu \right| : f : E \rightarrow \mathbb{C} \text{ measurable and } |f| \leq 1 \right\} \quad (3.70)$$

(compare (Halmos, 1950, p. 124)). Show⁵⁹ that

$$F(x) = \sum_{\alpha \in \mathbb{Z}_+^4} D_x^\alpha \mu_\alpha(x)$$

converges in the weak topology of $J^s(\mathbb{R}^4)'$ and that

$$M = \bigcap_{\alpha \in \mathbb{Z}_+^4} \text{supp } \mu_\alpha$$

is a quasi-support of F with respect to $J^s(\mathbb{R}^4)$.

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⁵⁹See (Lücke, 1984) for a proof of the converse of the corresponding statement for $S^s(\mathbb{R}^4)$.

In the nonlocalizable case, a useful substitute of Theorem 3.3.1 is the following.

Theorem 3.3.2 *Let $s > 0$, let $\mathcal{L}(p + iq)$ be an analytic function on $\mathbb{R}^4 + iV_+$, and assume that*

$$\sup_{p+iq \in \mathbb{R}^4 + iV_+} (\|q\| \operatorname{dist}(q, \partial V_+))^A e^{-\left(\frac{\|p\| + \|q\|}{A}\right)^{1/s}} |\mathcal{L}(p + iq)| < \infty$$

holds for sufficiently large $A > 0$. Then

$$\tilde{F}(p) = \lim_{\epsilon \rightarrow +0} \mathcal{L}(p + iq)$$

converges in the weak topology of $\widetilde{J^s}(\mathbb{R}^4)'$. Moreover, every closed cone with apex at $x = 0$ and containing $\overline{V_+} \setminus \{0\}$ in its interior is a quasi-support⁶⁰ of $F(x)$ with respect to $J^s(\mathbb{R}^4)$.

Proof: See (Fainberg and Soloviev, 1992, Theorem 4). ■

Moreover the following result is useful:

Theorem 3.3.3 *Let⁶¹ $s > 0$ and let $\{x \in \mathbb{R}^4 : x^0 \geq 0\}$ be a quasi-support (w.r.t. $J^s(\mathbb{R}^4)$) of $F(x) \in J^s(\mathbb{R}^4)'$. If $F(x)$ is Lorentz invariant then also $\overline{V_+}$ is a quasi-support of $F(x)$.*

Proof: See (Bümmerstedt, 1976, Theorem 4.7). ■

Let $\mathcal{V}(t)$ be an entire analytic function of **order** $\frac{1}{2s}$, i.e.:

$$\exists \rho > 0 : \sup_{z \in \mathbb{C}^1} e^{-\rho|z|^{2s}} |\mathcal{V}(z)| < \infty. \quad (3.71)$$

Then, by Theorems 3.3.2 and 3.3.3,

$$\Delta_{\text{ret}}^{\mathcal{V}}(x) \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow +0} (2\pi)^{-4} \int \frac{\mathcal{V}\left(\frac{p^2}{m^2}\right) e^{-ipx}}{(p^0 + i\epsilon)^2 - \mathbf{p}^2 - m^2} dp \quad (3.72)$$

is an element of $J^s(\mathbb{R}^4)$ with quasi-support $\overline{V_+}$. If, in addition,

$$\mathcal{V}(1) = 1 \quad (3.73)$$

⁶⁰Presumably, $\overline{V_+}$ itself is a quasi-support of $F(x)$ — as obvious for $s > 1$. A generalization of Theorem 3.3.2 for $s = 0$ was proved in (Soloviev, 1997).

⁶¹Actually, Theorem 3.3.3 is also valid for $s = 0$.

then also

$$\begin{aligned}\Delta_{\text{F}}^{\mathcal{V}}(x) &\stackrel{\text{def}}{=} \Delta_{\text{ret}}^{\mathcal{V}}(x) - \Delta_{-}(x) \\ &= \lim_{\epsilon \rightarrow +0} (2\pi)^{-4} \int \frac{\mathcal{V}\left(\frac{p^2}{m^2}\right) e^{-ipx}}{p^2 - m^2 - i\epsilon} dp\end{aligned}\quad (3.74)$$

is a well defined element of $J^s(\mathbb{R}^4)$. If $\mathcal{V}(t)$ is even of sufficiently fast decrease, e.g.

$$\sup_{t \in \mathbb{R}^1} t^2 |\mathcal{V}(t)| < \infty, \quad (3.75)$$

then there is a ‘canonical’ definition for powers of the modified FEYNMAN propagator $\Delta_{\text{F}}^{\mathcal{V}}(x)$ using FEYNMAN *parameterization*

$$\begin{aligned}\frac{1}{c_1 \cdots c_n} &= (n-1)! \int_0^1 d\xi_1 \cdots \int_0^1 d\xi_n \frac{\delta(1 - \xi_1 - \dots - \xi_n)}{(c_1 \xi_1 + \dots + c_n \xi_n)^n} \\ &\quad \text{if } 1 + c_1 \xi_1 + \dots + c_n \xi_n \neq 0 \quad \forall \xi_1, \dots, \xi_n \geq 0\end{aligned}\quad (3.76)$$

(see (Alebastrov and Efimov, 1973, Sect. 4.3)).

Proof of (3.76):

$$\begin{aligned}\frac{1}{n! c_1 \cdots c_n} &= \int_0^\infty d\xi_1 \cdots \int_0^\infty d\xi_n \frac{1}{(1 + c_1 \xi_1 + \dots + c_n \xi_n)^{n+1}} \\ &= \int_0^\infty d\xi_1 \cdots \int_0^\infty d\xi_n \int_0^\infty d\lambda \frac{\delta(\lambda - \xi_1 - \dots - \xi_n)}{(1 + c_1 \xi_1 + \dots + c_n \xi_n)^{n+1}} \\ &= \int_0^\infty d\lambda \int_0^\infty d\xi'_1 \cdots \int_0^\infty d\xi'_n \frac{\delta((1 - \xi'_1 - \dots - \xi'_n)\lambda)}{(1 + (c_1 \xi'_1 + \dots + c_n \xi'_n)\lambda)^{n+1}} \lambda^n \\ &= \int_0^\infty d\xi'_1 \cdots \int_0^\infty d\xi'_n \int_0^\infty d\lambda \frac{\delta(1 - \xi'_1 - \dots - \xi'_n)}{(\lambda^{-1} + c_1 \xi'_1 + \dots + c_n \xi'_n)^{n+1}} \lambda^{-2} \\ &= \int_0^\infty d\xi'_1 \cdots \int_0^\infty d\xi'_n \int_0^\infty d\lambda' \frac{\delta(1 - \xi'_1 - \dots - \xi'_n)}{(\lambda' + c_1 \xi'_1 + \dots + c_n \xi'_n)^{n+1}} \\ &= \frac{1}{n} \int_0^1 d\xi'_1 \cdots \int_0^1 d\xi'_n \frac{\delta(1 - \xi'_1 - \dots - \xi'_n)}{(c_1 \xi'_1 + \dots + c_n \xi'_n)^n} \quad \blacksquare\end{aligned}$$

Therefore, if we denote by $T_{\mathcal{V}}$ the time-ordering resulting by replacing – after formal application of WICK’s theorem – the products of the ordinary FEYNMAN propagator $\Delta_{\text{F}}(x)$ by the ‘canonical’ products of the modified FEYNMAN propagator $\Delta_{\text{F}}^{\mathcal{V}}(x)$,

$$\hat{S}_0^{\mathcal{V}}(g) \stackrel{\text{def}}{=} T^{\mathcal{V}} \left(\exp \left(\int \hat{S}_1(x) g(x) dx \right) \right) \quad \text{for } g \in J^s(\mathbb{R}^4) \quad (3.77)$$

fulfills all the requirements of the generalized BOGOLIUBOV-SHIRKOV theory.⁶²

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⁶²Generalized BOGOLIUBOV-SHIRKOV causality can be proved by Theorem 3.3.2 and the ALEBASTROV-EFIMOV analysis of $\frac{\delta}{\delta g(x)} \left(\frac{\delta \hat{S}_0(g)}{\delta g(y)} \hat{S}_0(g)^* \right)_{|g=0}$ in (Alebastrov and Efimov, 1974, Sect. 7). Of course, counterterms have still to be added if the adiabatic limit $g \rightarrow 1$ is to exist.

Unfortunately, as a consequence of the PHRAGMÉN-LINDELÖF theorem (see, e.g., Theorem 4.1.1 of (Lücke, *ftm*)), (3.71) and (3.75) are incompatible with each other for $s \geq 1/2$. For $s < 1/2$ there are plenty of entire functions $\mathcal{V}(t)$ fulfilling both (3.71) and (3.75) (see (Gelfand and Schilow, 1962, Kap. IV §8)). This is why EFIMOV (see (König, 1993) and references given there) suggested to use test spaces of entire functions in order to ‘regularize’ the perturbative expansion of the S -matrix.

For $s = 1$ we may still have

$$\sup_{t \in \mathbb{R}_+} t^2 \mathcal{V}(-t) < \infty \quad (3.78)$$

in addition to (3.73). Take

$$\mathcal{V}(t) = \left(\frac{\sin \sqrt{-t}}{\sqrt{-t}} \right)^4,$$

for instance. Now, the expressions resulting from the ‘canonical’ definition of powers of $\Delta_{\mathbb{F}}^{\mathcal{V}}(x)$ – thanks to transition to Euclidean momenta – are well defined, if only (3.78) holds. EFIMOV used this for the **definition** of ‘regularized’ solutions of the generalized BOGOLIUBOV-SHIRKOV theory⁶³ for $J^1(\mathbb{R}^4)$ – motivated by suitable quantization of formally nonlocal field theories (Efimov, 1974).

⁶³For generalization to gauge theories see, e.g., (Moffat, 1990; Cornish, 1992) and references given there.

Chapter 4

Quantum Electrodynamics

4.1 The Free Electromagnetic Field Operators

4.1.1 WIGHTMAN Theory

Axioms

The WIGHTMAN *axioms* for the free electromagnetic field $\hat{F}^{\mu\nu}(x)$ follow from those for the neutral scalar field by straightforward adaption:

0. Assumptions of Relativistic Quantum Theory:

The same as in 2.2.1.

I. Assumptions about the Domain and Continuity of the Fields:

The field operators $\hat{F}^{\mu\nu}(x)$ are *hermitian operator-valued, tempered generalized functions with invariant domain* $D \subset \mathcal{H}$; i.e. **linear** mappings

$$\begin{aligned} \hat{F}^{\mu\nu} : \mathcal{S}(\mathbb{R}^4) &\longrightarrow L(D, D) \\ \varphi &\longmapsto \hat{F}^{\mu\nu}(\varphi) = \underbrace{\int \hat{F}^{\mu\nu}(x)\varphi(x) dx}_{\text{formal}} \end{aligned}$$

for which all the

$$\int \langle \Psi | \hat{F}^{\mu\nu}(x)\Psi \rangle \varphi(x) dx \stackrel{\text{def}}{=} \langle \Psi | \hat{F}^{\mu\nu}(\varphi)\Psi \rangle, \Psi \in D,$$

are **continuous** in $\varphi \in \mathcal{S}(\mathbb{R}^4)$, where D has to fulfill the following conditions for $\varphi \in \mathcal{S}(\mathbb{R}^4)$ and $(a, \Lambda) \in \mathcal{P}_+^\uparrow$:

$$\Omega \in D, \quad \hat{U}(a, \Lambda)D \subset D, \quad \hat{F}^{\mu\nu}(\varphi)D \subset D, \quad \hat{F}^{\mu\nu}(\bar{\varphi}) = \hat{F}^{\mu\nu}(\varphi)^* \wedge D.$$

II. Transformation Law of the Fields:

The fields transform according to

$$\hat{U}(a, \Lambda)^{-1} \hat{F}^{\mu\nu}(x) \hat{U}(a, \Lambda) = \Lambda_{\alpha}^{\mu} \Lambda_{\beta}^{\nu} \hat{F}^{\alpha\beta}(\Lambda^{-1}(x - a)) \quad \forall (a, \Lambda) \in \mathcal{P}_+^{\uparrow}.$$

Remark: Recall that $\hat{U}(a, \Lambda)\Psi$ is to be described w.r.t. the coordinates $x' = \Lambda^{-1}(x - y)$ in exactly the same way as Ψ is to be described w.r.t. the coordinates x and that the field expectation values should transform like the classical fields (see, e.g., Eq. (2.9) of (Lücke, edyn)). Therefore:

$$\left\langle \hat{U}(a, \Lambda)\Psi \mid \hat{F}^{\mu\nu}(\Lambda x + a) \hat{U}(a, \Lambda) \right\rangle = \Lambda_{\alpha}^{\mu} \Lambda_{\beta}^{\nu} \left\langle \Psi \mid \hat{F}^{\alpha\beta}(x) \Psi \right\rangle.$$

III. Local Commutativity (Microscopic Causality):

The smeared fields $\hat{F}^{\mu\nu}(\varphi_1)$, $\hat{F}^{\alpha\beta}(\varphi_2)$ commute whenever the supports of the test functions $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^4)$ are spacelike with respect to each other. Formally:

$$x \times y \implies [\hat{F}^{\mu\nu}(x), \hat{F}^{\alpha\beta}(y)]_- = 0.$$

Again, the vacuum vector Ω is required to be **cyclic** with respect to the algebra \mathcal{F}_0 generated by $\hat{1} \wedge D$ and the smeared field operators $\hat{F}^{\mu\nu}(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^4)$, $\mu, \nu \in \{0, 1, 2, 3\}$. Finally, the field operators have to fulfill the free MAXWELL equations:¹

$$\partial_{\nu} \hat{F}^{\mu\nu}(x) = 0, \quad (4.1)$$

$$\partial^{\nu} \epsilon_{\mu\nu\alpha\beta} \hat{F}^{\alpha\beta}(x) = 0. \quad (4.2)$$

Essential Uniqueness of the WIGHTMAN Theory

Using the results of (Oksak and Todorov, 1969) and (Pohlmeyer, 1969) one can prove² the following variant of the Jost-Schroer theorem (Theorem 2.2.18):

Theorem 4.1.1 *The WIGHTMAN theory of the free electromagnetic field, as described above³ is unique up to unitary equivalence and up to some common constant factor of the field operators, if D is chosen to be the smallest linear subspace of \mathcal{H} containing Ω and being invariant under all the smeared field operators.*

A realization of the WIGHTMAN theory of the free electromagnetic field will be given in 4.1.3.

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¹As usual, we define

$$\epsilon_{\mu\nu\alpha\beta} \stackrel{\text{def}}{=} \begin{cases} +1 & \text{if } (\mu, \nu, \alpha, \beta) \text{ is an even permutation of } (0, 1, 2, 3), \\ -1 & \text{if } (\mu, \nu, \alpha, \beta) \text{ is an odd permutation of } (0, 1, 2, 3), \\ 0 & \text{else.} \end{cases}$$

²See also (Fredenhagen, 2001, III.4).

³As pointed out by D. Buchholz (private communication) it is sufficient to require the validity of the wave equation for every component of the field tensor rather than the full set of MAXWELL equations.

Vacuum Fluctuations

Although we will see that

$$\langle \Omega | \hat{F}^{\mu\nu}(x)\Omega \rangle = 0,$$

as to be expected, cyclicity of Ω implies that the 2-point functions

$$\langle \Omega | \hat{F}^{\mu\nu}(x)\hat{F}^{\mu\nu}(y)\Omega \rangle$$

cannot all vanish.⁴ Therefore the variance of the field strength is nonzero already in the vacuum state. This is due to *vacuum fluctuations*, which also cause *spontaneous emission* of photons from atoms in excited states (see, e.g. (Baym, 1969, S. 278 ff.)) and may be experimentally verified by the so-called *Casimir effect* (see, e.g. (Itzykson and Zuber, 1980a, Section 3-2-4)).

The permanent presence of perturbations like those connected with vacuum fluctuations is one of the main problems of quantum field theory.

4.1.2 Problems With the Quantized Potentials

Desirable Properties

Just as in classical electrodynamics it is convenient – and for coupling to the charged matter fields also necessary – to introduce potentials $\hat{A}^\mu(x)$ which we should like to fulfill the following requirements:⁵

- (i) The $\hat{A}(\mathbf{x}, t)$ are operator-valued tempered generalized functions with invariant dense domain $D_A \subset \mathcal{H}$.
- (ii) There is a nondegenerate⁶ continuous sesquilinear form $(. | .)$ on \mathcal{H} w.r.t. which

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⁴This fact implies interesting restrictions on the joint measurability of different field components (Bohr and Rosenfeld, 1933) – a necessary supplement to the Heisenberg uncertainty relations of ordinary quantum mechanics.

⁵See also (Strocchi, 1977) and (Lücke, edyn, Sect. 2.3.2). Of course, it would be nice to have $D_A = D_F = D$ and $(. | .) = \langle . | . \rangle$. By Strocchi's Theorem (Theorem 4.1.2), however, this is not possible. While it is easy to see that quantized potentials can always be constructed in some bigger space with indefinite metric – even in the noninteracting case (Bongaarts, 1977) – there are almost no physical hints on which properties can be expected for such auxiliary field operators.

⁶In general a quadratic form $(. | .)$ on $\mathcal{H} \times \mathcal{H}$ is called *nondegenerate* if the mapping

$$\mathcal{H} \ni \Psi \mapsto (\Psi | .) \in \mathcal{H}'$$

is a bijection. For Hilbert spaces \mathcal{H} this is equivalent to:

$$(\Phi | \Phi') = 0 \forall \Phi' \in \mathcal{H} \implies \Phi = 0.$$

the $\hat{\mathbf{A}}(\mathbf{x}, t)$ are hermitian:⁷

$$\left(\Phi_1 \mid \hat{\mathbf{A}}(\mathbf{x}, t) \Phi_2 \right) = \left(\hat{\mathbf{A}}(\mathbf{x}, t) \Phi_1 \mid \Phi_2 \right) \quad \forall \Phi_1, \Phi_2 \in D_A. \quad (4.3)$$

(iii) There is a representation $\hat{V}(a, \Lambda)$ of \mathcal{P}_+^\uparrow with

$$D_{\hat{V}(a, \Lambda)} = R_{\hat{V}(a, \Lambda)} = D_A \quad (4.4)$$

which is strongly continuous w.r.t. $\langle \cdot \mid \cdot \rangle$, unitary w.r.t. $(\cdot \mid \cdot)$, i.e.

$$\left(\Phi_1 \mid \hat{V}(a, \Lambda) \Phi_2 \right) = \left(\hat{V}(a, \Lambda)^{-1} \Phi_1 \mid \Phi_2 \right) \quad \forall \Phi_1, \Phi_2 \in D_A, \quad (4.5)$$

and transforming the $\hat{\mathbf{A}}(\mathbf{x}, t)$ according to

$$\hat{V}(a, \Lambda)^{-1} \hat{\mathbf{A}}(\mathbf{x}, t) \hat{V}(a, \Lambda) = (\Lambda_A)^\mu{}_\nu \hat{A}^\nu (\Lambda_A^{-1}(x - a)). \quad (4.6)$$

(iv) There is a vector $\Omega \in D_A$, **unique** up to a phase factor, fulfilling

$$(\Omega \mid \Omega) = 1 \quad \text{and} \quad \hat{V}(a, \Lambda) \Omega = \Omega \quad \forall (a, \Lambda) \in \mathcal{P}_+^\uparrow. \quad (4.7)$$

(v)

$$x \times y \implies \left[\hat{\mathbf{A}}(\mathbf{x}, t), \hat{A}^\nu(y) \right]_- = 0. \quad (4.8)$$

(vi)

$$\square \hat{A}^\nu(x) = 0 \quad \forall \nu \in \{0, \dots, 3\}. \quad (4.9)$$

(vii) There is a linear subspace D_F of D_A fulfilling the conditions

$$\left(\Phi_1 \mid \partial_\mu \hat{\mathbf{A}}(\mathbf{x}, t) \Phi_2 \right) = 0 \quad \forall \Phi_1, \Phi_2 \in D_F, \quad (4.10)$$

$$\hat{F}_{\hat{\mathbf{A}}}^{\mu\nu}(x) D_F \subset D_F \supset \hat{V}(a, \Lambda) D_F \quad \forall \mu, \nu \in \{0, \dots, 3\}, (a, \Lambda) \in \mathcal{P}_+^\uparrow, \quad (4.11)$$

and

$$\Omega \in D_F \quad (\subset D_A \subset \mathcal{H}). \quad (4.12)$$

where

$$\hat{F}_{\hat{\mathbf{A}}}^{\mu\nu}(x) \stackrel{\text{def}}{=} \partial^\mu \hat{A}^\nu(x) - \partial^\nu \hat{\mathbf{A}}(\mathbf{x}, t), \quad (4.13)$$

(viii)

$$\int \left(\Phi_1 \mid \hat{V}(a, \mathbb{1}_4) \Phi_2 \right) e^{-ip^\mu a_\mu} da = 0 \quad \forall p \in \mathbb{R}^4 \setminus \overline{V_+}, \Phi_1, \Phi_2 \in D_A. \quad (4.14)$$

⁷Of course, we should like to take $(\cdot \mid \cdot) = \langle \cdot \mid \cdot \rangle$. However, as will be shown in Corollary 4.1.4, this would be in contradiction to the other assumptions made below.

(ix)

$$\overline{\mathcal{Z}_A^{(\cdot|\cdot)}} = \mathcal{H},$$

where \mathcal{Z}_A denotes the smallest linear subspace of D_A containing Ω and being invariant under all $\hat{\mathbf{A}}(\mathbf{x}, t)$.

(x) For every $\Phi \in D_F$ and every $\epsilon > 0$ there is a $\Phi' \in \mathcal{Z}_F$ with

$$(\Phi - \Phi' | \Phi - \Phi') < \epsilon,$$

where \mathcal{Z}_F denotes the smallest linear subspace of D_F containing Ω and being invariant under all $\hat{F}_{\hat{A}}^{\mu\nu}(x)$.

If $(\cdot | \cdot)$ were positive definite,

$$\mathcal{H}_F \stackrel{\text{def}}{=} \overline{D_F^{(\cdot|\cdot)}}, \overline{\hat{V}(a, \Lambda)^{(\cdot|\cdot)}}, \Omega \text{ and } \hat{F}^{\mu\nu}$$

would give a WIGHTMAN theory of the free electromagnetic field.

No-Go Theorem

Theorem 4.1.2 (Strocchi) *Unless all $\hat{F}_{\hat{A}}^{\mu\nu}(x)$ vanish, conditions (i)–(x) imply*

$$\boxed{\partial_\mu \hat{F}_{\hat{A}}^{\mu\nu}(x) \Omega \neq 0.} \quad (4.15)$$

Proof: See (Strocchi, 1970). ■

Lemma 4.1.3 *Let D be a (complex) linear space with positive semi-definite sesquilinear form $(\cdot | \cdot)$ and let $\hat{B}_1, \hat{B}_2 \in L(D, D)$ fulfill the condition*

$$(\Phi' | \hat{B}_1 \Phi) = (\hat{B}_2 \Phi' | \Phi) \quad \forall \Phi, \Phi' \in D.$$

Then

$$\hat{B}_1 D_{00} \subset D_{00} \stackrel{\text{def}}{=} \{\Phi \in D : (\Phi | \Phi) = 0\}.$$

Proof: By means of **Schwartz'** inequality (see (Strocchi and Wightman, 1974); Lemma 2.2 and application). ■

Corollary 4.1.4 *Unless all $\hat{F}_{\hat{A}}^{\mu\nu}(x)$ vanish, conditions (i)–(x) imply that $(\cdot | \cdot)$ can be neither⁸ positive definite on D_F nor positive semi-definite on D_A if*

$$(\Phi | \partial_\mu \hat{F}_{\hat{A}}^{\mu\nu}(x) \Phi) = 0 \quad \forall \Phi \in D_F.$$

⁸By (4.9)–(4.13) positive definiteness on D_F would imply $\partial_\mu \hat{F}_{\hat{A}}^{\mu\nu}(x) = 0$, in contradiction to Theorem 4.1.2. The case of positive semi-definiteness on D_A may be reduced to the case of positive definiteness via factorization, by Lemma 4.1.3.

GUPTA-BLEULER Generalization

Gupta (Gupta, 1950) and Bleuler (Bleuler, 1950) realized that, fortunately, it is sufficient to postulate positive semi-definiteness of $(\cdot | \cdot)$ on D_F :

$$(\Phi | \Phi) \geq 0 \quad \forall \Phi \in D_F. \quad (4.16)$$

Then assumptions (i)–(ix) guarantee that factorization⁹ of $D_F, \hat{V}(a, \Lambda) \wedge D_F$ and $\hat{F}_{\hat{A}}^{\mu\nu}(x) \wedge D_F$ by

$$D_{00} \stackrel{\text{def}}{=} \{ \Phi \in D_F : (\Phi | \Phi) = 0 \}, \quad (4.17)$$

gives a WIGHTMAN theory of the free electromagnetic field (compare last part of 4.1.3).

4.1.3 GUPTA-BLEULER Construction

Quantized Electromagnetic Potentials

As domain D_A for the quantized potentials we choose the set of all truncated sequences

$$A_{\text{GB}} = \{A_0, A_1, \dots, A_n, 0, 0, \dots\},$$

where

$$A_0 \in \mathbb{C}, \quad A_n \hat{=} \{a^{\mu_1, \dots, \mu_n}(\mathbf{p}_1, \dots, \mathbf{p}_n)\}_{\mu_j=0,1,2,3},$$

and

$$a^{\mu_1, \dots, \mu_n}(\mathbf{p}_1, \dots, \mathbf{p}_n) = a^{\mu_{\pi_1}, \dots, \mu_{\pi_n}}(\mathbf{p}_{\pi_1}, \dots, \mathbf{p}_{\pi_n}) \in \mathcal{S}(\mathbb{R}^{3n}) \quad \forall \pi \in S_n. \quad (4.18)$$

As inner product we choose

$$\begin{aligned} & \langle A_{\text{GB}} | \check{A}_{\text{GB}} \rangle \\ & \stackrel{\text{def}}{=} \overline{A_0} \check{A}_0 + \sum_{n=1}^{\infty} \sum_{\mu_1, \dots, \mu_n=0}^3 \int \overline{a^{\mu_1, \dots, \mu_n}(\mathbf{p}_1, \dots, \mathbf{p}_n)} \check{a}^{\mu_1, \dots, \mu_n}(\mathbf{p}_1, \dots, \mathbf{p}_n) \frac{d\mathbf{p}_1}{2|\mathbf{p}_1|} \cdots \frac{d\mathbf{p}_n}{2|\mathbf{p}_n|}. \end{aligned} \quad (4.19)$$

Then \mathcal{H} is chosen to be the completion of D_A with respect to this inner product. Similarly to (2.23) resp. (2.32) we define annihilation operators $\hat{a}^\mu(\mathbf{p})$ by¹⁰

$$\begin{aligned} & (\hat{a}^\mu(\mathbf{p}) A_{\text{GB}})_0 \stackrel{\text{def}}{=} a^\mu(\mathbf{p}) \\ & (\hat{a}^\mu(\mathbf{p}) A_{\text{GB}})^{\mu_1, \dots, \mu_{n-1}}(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}) \stackrel{\text{def}}{=} \sqrt{n} a^{\mu, \mu_1, \dots, \mu_{n-1}}(\mathbf{p}, \mathbf{p}_1, \dots, \mathbf{p}_{n-1}) \quad \text{for } n > 1 \end{aligned} \quad (4.20)$$

⁹Factorization is possible thanks to Lemma 4.1.3.

¹⁰We use the identification $(A_{\text{GB}})^{\mu_1, \dots, \mu_\nu}(\mathbf{p}_1, \dots, \mathbf{p}_\nu) = a^{\mu_1, \dots, \mu_\nu}(\mathbf{p}_1, \dots, \mathbf{p}_\nu)$.

resp. creation operators $\hat{a}^\mu(\mathbf{p})^\dagger$ by¹¹

$$\begin{aligned} (\hat{a}^\mu(\mathbf{p})^\dagger A_{\text{GB}})_0 &\stackrel{\text{def}}{=} 0, \\ (\hat{a}^\mu(\mathbf{p})^\dagger A_{\text{GB}})^{\mu_1}(\mathbf{p}_1) &\stackrel{\text{def}}{=} -2|\mathbf{p}_1| g^{\mu\mu_1} \delta(\mathbf{p} - \mathbf{p}_1) A_0, \\ (\hat{a}^\mu(\mathbf{p})^\dagger A_{\text{GB}})^{\mu_1, \dots, \mu_{n+1}}(\mathbf{p}_1, \dots, \mathbf{p}_{n+1}) &\stackrel{\text{def}}{=} -\sqrt{\frac{1}{n+1}} \sum_{j=1}^{n+1} 2|\mathbf{p}_j| g^{\mu\mu_j} \delta(\mathbf{p} - \mathbf{p}_j) \cdot \\ &\quad \cdot a^{\mu_1, \dots, \mu_{\hat{j}}, \dots, \mu_{n+1}}(\mathbf{p}_1, \dots, \mathbf{p}_{\hat{j}}, \dots, \mathbf{p}_{n+1}). \end{aligned} \quad (4.21)$$

Then, thanks to the commutation relations¹²

$$\begin{aligned} [\hat{a}^\mu(\mathbf{p}), \hat{a}^{\mu'}(\mathbf{p}')^\dagger]_- &= -2|\mathbf{p}| g^{\mu\mu'} \delta(\mathbf{p} - \mathbf{p}'), \\ [\hat{a}^\mu(\mathbf{p}), \hat{a}^{\mu'}(\mathbf{p}')]_- &= [\hat{a}^\mu(\mathbf{p})^\dagger, \hat{a}^{\mu'}(\mathbf{p}')^\dagger]_- = 0 \end{aligned} \quad (4.22)$$

the quantized potentials¹³

$$\hat{A}_{\text{GB}}^\mu(x) \stackrel{\text{def}}{=} \hat{A}_{\text{GB}}^{\mu(+)}(x) + \hat{A}_{\text{GB}}^{\mu(-)}(x), \quad (4.23)$$

$$\hat{A}_{\text{GB}}^{\mu(+)}(x) \stackrel{\text{def}}{=} \sqrt{\zeta} (2\pi)^{-3/2} \int_{p_0=+|\mathbf{p}|} \hat{a}^\mu(\mathbf{p}) e^{-\frac{i}{\hbar} p^\nu x_\nu} \frac{d\mathbf{p}}{2|\mathbf{p}|}, \quad (4.24)$$

$$\hat{A}_{\text{GB}}^{\mu(-)}(x) \stackrel{\text{def}}{=} \sqrt{\zeta} (2\pi)^{-3/2} \int_{p_0=-|\mathbf{p}|} \hat{a}^\mu(-\mathbf{p})^\dagger e^{-\frac{i}{\hbar} p^\nu x_\nu} \frac{d\mathbf{p}}{2|\mathbf{p}|}, \quad (4.25)$$

obeying condition (i) of 4.1.2, fulfill the commutation relations¹⁴

$$\left[\hat{A}_{\text{GB}}^{\mu(+)}(x), \hat{A}_{\text{GB}}^{\mu'(-)}(x') \right]_- = -\zeta g^{\mu\mu'} i \Delta_0^{(+)}(x - x'), \quad (4.26)$$

$$\left[\hat{A}_{\text{GB}}^{\mu(+)}(x), \hat{A}_{\text{GB}}^{\mu'(+)}(x') \right]_- = \left[\hat{A}_{\text{GB}}^{\mu(-)}(x), \hat{A}_{\text{GB}}^{\mu'(-)}(x') \right]_- = 0,$$

$$\left[\hat{A}_{\text{GB}}^\mu(x), \hat{A}_{\text{GB}}^{\mu'}(x') \right]_- = -\zeta g^{\mu\mu'} i \Delta_0(x - x'). \quad (4.27)$$

(4.27) directly implies condition (v) of 4.1.2. By (4.23)–(4.25) also condition (vi) of 4.1.2 is fulfilled. According to (4.20)–(4.25) we **have to** define

$$(A_{\text{GB}} | A'_{\text{GB}}) \stackrel{\text{def}}{=} \langle A_{\text{GB}} | \hat{\eta} A'_{\text{GB}} \rangle, \quad (4.28)$$

where

$$\begin{aligned} (\hat{\eta} A_{\text{GB}})^{\mu_1, \dots, \mu_n}(\mathbf{p}_1, \dots, \mathbf{p}_n) &\stackrel{\text{def}}{=} (-g_{\mu_1 \nu_1}) \cdots (-g_{\mu_n \nu_n}) a^{\nu_1, \dots, \nu_n}(\mathbf{p}_1, \dots, \mathbf{p}_n), \\ (\hat{\eta} A_{\text{GB}})_0 &\stackrel{\text{def}}{=} A_0, \end{aligned} \quad (4.29)$$

in order to fulfill (4.3) and $(\Omega | \Omega) = 1$ for the vector

$$\Omega \stackrel{\text{def}}{=} \{1, 0, 0, \dots\}. \quad (4.30)$$

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¹¹The use of $-g^{\mu\mu_1}$ instead of $\delta_{\mu\mu_1}$ is necessary to yield the vector transformation property (4.6) with (4.31) for the GUPTA-BLEULER potentials $\hat{A}^\mu(x) = \hat{A}_{\text{GB}}^\mu(x)$.

¹²Compare (2.34).

¹³Compare (2.39), (2.40), and (2.43). For the physically correct choice of ζ see Exercise 61.

¹⁴Compare (2.46), (2.47), and (2.62).

Since $\hat{\eta}$ is unitary¹⁵ w.r.t. $\langle \cdot | \cdot \rangle$ the sesquilinear form $(\cdot | \cdot)$, defined this way, fulfills condition (ii) of 4.1.2, indeed. By

$$\begin{aligned} & \left(\hat{V}(a, \Lambda) A_{\text{GB}} \right)_0 \stackrel{\text{def}}{=} A_0, \\ & \left(\hat{V}(a, \Lambda) A_{\text{GB}} \right)^{\mu_1, \dots, \mu_n}(\mathbf{p}_1, \dots, \mathbf{p}_n) \\ & \stackrel{\text{def}}{=} e^{i(p_1 + \dots + p_n)a} \Lambda^{\mu_1}_{\nu_1} \dots \Lambda^{\mu_n}_{\nu_n} a^{\nu_1, \dots, \nu_n} \left(\overrightarrow{\Lambda^{-1} p_1}, \dots, \overrightarrow{\Lambda^{-1} p_n} \right) \Big|_{p_j^0 = |\mathbf{p}_j|} \end{aligned} \quad (4.31)$$

(compare (2.21)), since (4.19) and (4.28) imply

$$(A_{\text{GB}} | A'_{\text{GB}}) = \overline{A_0} A'_0 + \sum_{n=1}^{\infty} \int \overline{a^{\mu_1, \dots, \mu_n}(\mathbf{p}_1, \dots, \mathbf{p}_n)} a'_{\mu_1, \dots, \mu_n}(\mathbf{p}_1, \dots, \mathbf{p}_n) \frac{d\mathbf{p}_1}{2|\mathbf{p}_1|} \dots \frac{d\mathbf{p}_n}{2|\mathbf{p}_n|}, \quad (4.32)$$

we get a representation of \mathcal{P}_+^\dagger fulfilling conditions (iii), (iv), and (viii) of 4.1.2.

Warning: For $\mathbf{v} \neq 0$, the operators $\hat{V}(\Lambda_{\mathbf{v}}, 0)$ are **unbounded** w.r.t. $\langle \cdot | \cdot \rangle$.

Proof For every $\mathbf{v} \neq 0$ there is a 1-photon state vector \hat{A}_{GB} fulfilling

$$\begin{aligned} & \langle \hat{A}_{\text{GB}} | \hat{A}_{\text{GB}} \rangle = 1, \\ & \left\| \hat{V}(\Lambda_{\mathbf{v}}, 0) A_{\text{GB}} \right\|^2 \stackrel{\text{def}}{=} \langle \hat{V}(\Lambda_{\mathbf{v}}, 0) \hat{A}_{\text{GB}} | \hat{V}(\Lambda_{\mathbf{v}}, 0) \hat{A}_{\text{GB}} \rangle = 1 + \epsilon, \quad \epsilon > 0. \end{aligned}$$

Then, for

$$\begin{aligned} & (A_{\text{GB}}^N)^{\mu_1, \dots, \mu_n}(\mathbf{p}_1, \dots, \mathbf{p}_n) \\ & \stackrel{\text{def}}{=} \begin{cases} \sqrt{\frac{\epsilon}{N} \frac{1}{(1+\epsilon)^{n+1}}} \hat{a}^{\mu_1}(\mathbf{p}_1) \dots \hat{a}^{\mu_n}(\mathbf{p}_n) & \text{if } n < 1 + N^2 \frac{1+\epsilon}{\epsilon}, \\ 0 & \text{else,} \end{cases} \end{aligned}$$

we have¹⁶

$$\|A_{\text{GB}}^N\|^2 < \frac{1}{N}, \quad \left\| \hat{V}(\Lambda_{\mathbf{v}}, 0) A_{\text{GB}}^N \right\|^2 > N. \quad \blacksquare$$

Field Operators

In view of (4.15) the simplest possibility to fulfill also condition (vii) of 4.1.2 is¹⁷

$$D_F \stackrel{\text{def}}{=} \left\{ \Phi \in D_A : \partial_\mu \hat{A}_{\text{GB}}^{\mu(+)}(x) \Phi = 0 \right\}. \quad (4.33)$$

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¹⁵W.r.t. $(\cdot | \cdot)$, of course, $\hat{\eta}$ is unbounded.

¹⁶Note that $\sum_{n=0}^{\infty} \frac{\epsilon}{(1+\epsilon)^{n+1}} = 1$.

¹⁷Then $\partial_\mu \hat{A}_{\text{GB}}^{\mu}(\varphi) D_F \subset D_{00} \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^4)$.

Then¹⁸

$$D_F = D_{\text{Coul}} + D_{00}, \text{ where} \quad (4.34)$$

$$D_{\text{Coul}} \stackrel{\text{def}}{=} \left\{ \Phi \in D_F : \hat{A}_{\text{GB}}^{(+)\,0}(x)\Phi = 0 \right\},$$

and

$$(\Phi | \Phi') = 0 \quad \forall \Phi \in D_F, \Phi' \in D_{00}. \quad (4.35)$$

Since $(. | .)$ and $\langle . | . \rangle$ coincide on D_{Coul} this shows that $(. | .)$ is positive semi-definite on D_F .

Sketch of proof for (4.34):

$$A_{\text{GB}} - \hat{P}_{\text{Coul}} A_{\text{GB}} \in D_{00} \quad \text{with} \quad \hat{P}_{\text{Coul}} = \prod_{j=1}^{\infty} (1 - \hat{T}_j),$$

where

$$\left(\hat{T}_j A_{\text{GB}} \right)^{\mu_1, \dots, \mu_n}(\mathbf{p}_1, \dots, \mathbf{p}_n)$$

$$\stackrel{\text{def}}{=} \begin{cases} \frac{p_j^{\mu_j}}{p_j^0} a^{\mu_1, \dots, \overbrace{0}^{j. \text{ place}}, \dots, \mu_n}(\mathbf{p}_1, \dots, \mathbf{p}_n) \Big|_{p_j^0 = |\mathbf{p}_j|} & \text{for } j = 1, \dots, n, \\ 0 & \text{for } j > n. \quad \blacksquare \end{cases}$$

Sketch of proof for (4.35):

$$\begin{aligned} \langle A_{\text{GB}} | A_{\text{GB}} \rangle &= \left(\hat{P}_{\text{Coul}} A_{\text{GB}} | \hat{P}_{\text{Coul}} A_{\text{GB}} \right) \quad \forall A_{\text{GB}} \in D_F, \\ \rightsquigarrow A_{\text{GB}} \in D_{00} &\iff A_{\text{GB}} = (1 - \hat{P}_{\text{Coul}}) A_{\text{GB}}. \quad \blacksquare \end{aligned}$$

Exercise 60 Show that $\partial_\mu \hat{A}_{\text{GB}}^{\mu(+)}(x)\Phi$ commutes with all the smeared field operators $\hat{F}_{\text{BG}}^{\alpha\beta}(x)$ on D_A and hence the latter leave D_F invariant.

One easily checks¹⁹ that for every $\Phi \in D_{\text{Coul}}$ and for every $\epsilon > 0$ there is a $\Phi' \in \mathcal{Z}_F^{\text{Coul}}$ with $(\Phi - \Phi' | \Phi - \Phi') < \epsilon$, where $\mathcal{Z}_F^{\text{Coul}}$ denotes the smallest linear subspace of D_{Coul} containing Ω and being invariant under all $\hat{F}_{\text{GB}}^{jk}(x)$ with $j, k \in \{1, 2, 3\}$. This is because in D_{Coul} the topology induced by $(. | .)$ is equivalent to that induced by

$$\| \| A_{\text{GB}} \| \|^2 \stackrel{\text{def}}{=} |A_0|^2 + \sum_{n=1}^{\infty} \sum_{\mu_1, \dots, \mu_n=1}^3 \int |a^{\mu_1, \dots, \mu_n}(\mathbf{p}_1, \dots, \mathbf{p}_n)|^2 \frac{d\mathbf{p}_1}{2|\mathbf{p}_1|} \cdots \frac{d\mathbf{p}_n}{2|\mathbf{p}_n|}.$$

Therefore (4.34)/(4.35) imply that also condition (x) of 4.1.2 is fulfilled for $\hat{F}_{\text{GB}}^{\mu\nu}(x)$.

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¹⁸ D_{00} was defined in (4.17).

¹⁹Note that

$$\mathbf{p} \cdot \mathbf{a} \implies \mathbf{a} = \mathbf{p} \times \frac{\mathbf{a} \times \mathbf{p}}{|\mathbf{p}|^2}.$$

Transition to the WIGHTMAN Theory

Let D' denote the set of all equivalence classes

$$D' \stackrel{\text{def}}{=} \{[A_{\text{GB}}] : A_{\text{GB}} \in D_F\} \quad (4.36)$$

of D_F corresponding to the equivalence relation

$$A_{\text{GB}} \sim A'_{\text{GB}} \iff A_{\text{GB}} - A'_{\text{GB}} \in D_{00}. \quad (4.37)$$

Then, according to (4.35),

$$\langle [A_{\text{GB}}] | [A'_{\text{GB}}] \rangle \stackrel{\text{def}}{=} (A_{\text{GB}} | A'_{\text{GB}}) \quad (4.38)$$

does not depend on the choice of representatives $A_{\text{GB}}, A'_{\text{GB}} \in D_F$ and, by condition (ii) of 4.1.2 and (4.34)/(4.35), and defines a positive definite inner product on D' . Hence, the completion \mathcal{H}' of D' w.r.t. $\langle \cdot | \cdot \rangle$ is a **Hilbert** space. The appropriate representation of \mathcal{P}_+^\uparrow results from continuous extension of the operators²⁰

$$\hat{V}'(a, \Lambda) [A_{\text{GB}}] \stackrel{\text{def}}{=} [\hat{V}(a, \Lambda) A_{\text{GB}}] \quad (4.39)$$

onto all of \mathcal{H}' . Then, up to a constant factor,

$$\Omega' \stackrel{\text{def}}{=} [\Omega] \quad (4.40)$$

is the only element of \mathcal{H}' that is invariant under all $\hat{V}'(a, \Lambda)$. The spectrum condition follows from condition (viii) of 4.1.2. To summarize:

For \mathcal{H}' , $\hat{V}'(a, \Lambda)$, and Ω' all requirements of the zeroth WIGHTMAN axiom are fulfilled.

Next, we define the field operators:²¹

$$\hat{F}'^{\mu\nu}(\varphi)\Phi \stackrel{\text{def}}{=} \left[\hat{F}_{\hat{A}_{\text{GB}}}^{\mu\nu}(\varphi) A_{\text{GB}} \right] \text{ for } \Phi = [A_{\text{GB}}] \in D'. \quad (4.41)$$

Now all the WIGHTMAN axioms for the free electromagnetic field are fulfilled, as may be easily derived from conditions (i)–(x) of 4.1.2.

Corollary 4.1.5 *There is a WIGHTMAN theory of the free electromagnetic field, given by \mathcal{H}' , $\hat{V}'(a, \Lambda)$, Ω' , and $\hat{F}'^{\mu\nu}(x)$, with*

$$\left\langle \Omega' | \hat{F}'^{\mu_1\nu_1}(x_1) \cdots \hat{F}'^{\mu_n\nu_n}(x_n) \Omega' \right\rangle = \left(\Omega | \hat{F}_{\hat{A}_{\text{GB}}}^{\mu_1\nu_1}(x_1) \cdots \hat{F}_{\hat{A}_{\text{GB}}}^{\mu_n\nu_n}(x_n) \Omega \right)$$

for all $\mu_1, \nu_1, \dots, \mu_n, \nu_n \in \{0, \dots, 3\}$.

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²⁰Definition (4.39) is allowed by (4.11), (4.5) and Lemma 4.1.3.

²¹By (4.11), (4.3), and Lemma 4.1.3 this definition is allowed. The $\hat{F}_{\hat{A}_{\text{GB}}}^{\mu\nu}(x)$ were defined in (4.13).

Final remark: $\partial_\mu \hat{A}_{\text{GB}}^{\mu(-)}(x) = 0$ does **not** hold on all of D_F . Hence

$$\partial \hat{F}_{\hat{A}_{\text{GB}}}^{\mu\nu}(x) \neq 0$$

(compare Theorem 4.1.2). However, the following statements are true:

$$\partial^\nu \epsilon_{\mu\nu\alpha\beta} \hat{F}_{\hat{A}_{\text{GB}}}^{\alpha\beta}(x) \stackrel{(4.13)}{=} 0 \quad \text{on } D_A,$$

$$\partial_\mu \hat{F}_{\hat{A}_{\text{GB}}}^{\mu\nu}(x) \stackrel{(4.9)/(4.13)}{=} \partial^\nu \partial_\mu \hat{A}_{\text{GB}}^\mu(x),$$

$$\partial_\mu \left\langle A_{\text{GB}} \mid \hat{F}_{\hat{A}_{\text{GB}}}^{\mu\nu}(x) \mid A'_{\text{GB}} \right\rangle \stackrel{(4.10)}{=} 0 \quad \forall A_{\text{GB}}, A'_{\text{GB}} \in D_F.$$

4.1.4 GUPTA-BLEULER Observables

Agreement: By GUPTA-BLEULER *observable* we will always mean a $(\cdot \mid \cdot)$ -hermitian operator $\hat{A} \in L(D_A, D_A)$ leaving D_F invariant for which \hat{A}' , defined by

$$\hat{A}'[A_{\text{GB}}] \stackrel{\text{def}}{=} \left[\hat{A} A_{\text{GB}} \right] \quad \forall A_{\text{GB}} \in D_F$$

(recall Lemma 4.1.3), is an essentially selfadjoint operator of the WIGHTMAN theory constructed as described in 4.1.3. We will call GUPTA-BLEULER observables \hat{A} and \hat{B} *equivalent* if they induce the same transformation of the equivalence classes:

$$\hat{A} \sim \hat{B} \stackrel{\text{def}}{\iff} \left(\left[\hat{A} A_{\text{GB}} \right] = \left[\hat{B} A_{\text{GB}} \right] \quad \forall A_{\text{GB}} \in D_F \right).$$

Gauge Transformations

By *gauge transformation in*²² $\{\mathcal{H}, \langle \cdot \mid \cdot \rangle, \hat{\eta}, D_A, \Omega\}$ we will mean transition from one quantized potential $\hat{\mathbf{A}}(\mathbf{x}, t)$ to another quantized potential $\hat{A}'^\mu(x)$ without changing the n -point functions of the corresponding field strength operators:

$$\left(\Omega \mid \hat{F}_{\hat{A}}^{\mu_1\nu_1}(x_1) \cdots \hat{F}_{\hat{A}}^{\mu_n\nu_n}(x_n) \right) = \left(\Omega \mid \hat{F}_{\hat{A}'}^{\mu_1\nu_1}(x_1) \cdots \hat{F}_{\hat{A}'}^{\mu_n\nu_n}(x_n) \right).$$

Certainly, the latter is guaranteed if the connection between $\hat{\mathbf{A}}(\mathbf{x}, t)$ and $\hat{A}'^\mu(x)$ is given by $\hat{A}'^\mu(x) = \hat{\mathbf{A}}(\mathbf{x}, t) + \partial^\mu \hat{\chi}(x)$, where $\hat{\chi}(x)$ is a $(\cdot \mid \cdot)$ -hermitian tempered field in $\{\mathcal{H}, \langle \cdot \mid \cdot \rangle\}$ with invariant domain D_A . Not quite that simple is the gauge transformation

$$\hat{A}_{\text{GB}}^\mu(x) \longrightarrow \hat{A}_{\text{Coul}}^\mu(x) \stackrel{\text{def}}{=} \hat{A}_{\text{GB}}^\mu(x) + \hat{\chi}^\mu(x) \quad (4.42)$$

²²For a more general classification of gauge transformations see (Strocchi and Wightman, 1974).

with

$$\begin{aligned}\hat{\chi}^\mu(x) &\stackrel{\text{def}}{=} \sqrt{\zeta} (2\pi)^{-3/2} \int_{p^0=|\mathbf{p}|} (\hat{c}^\mu(\mathbf{p})e^{-ip^\mu x_\mu} + \hat{c}^\mu(\mathbf{p})^\dagger e^{+ip^\mu x_\mu}) \frac{d\mathbf{p}}{2p^0}, \\ \hat{c}^\mu(\mathbf{p}) &\stackrel{\text{def}}{=} \begin{cases} -\hat{a}^0(\mathbf{p}) & \text{for } \mu = 0, \\ -\frac{p^\mu}{|\mathbf{p}|^2} \sum_{j=1}^3 p^j \hat{a}^j(\mathbf{p}) & \text{for } \mu = 1, 2, 3. \end{cases}\end{aligned}\quad (4.43)$$

A simple calculation shows that

$$\hat{F}_{\hat{A}_{\text{GB}}}^{jk}(x) = \hat{F}_{\hat{A}_{\text{Coul}}}^{jk}(x) \quad \text{for } j, k \in \{1, 2, 3\}$$

and²³

$$\left(\hat{F}_{\hat{A}_{\text{GB}}}^{0j}(\varphi) - \hat{F}_{\hat{A}_{\text{Coul}}}^{0j}(\varphi) \right) D_F \subset D_{00} \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^4). \quad (4.44)$$

Hence, indeed, (4.42)/(4.43) is a gauge transformation. The index ‘‘Coul’’ is to indicate validity of the equations

$$\hat{A}_{\text{Coul}}^0(x) = 0, \quad (4.45)$$

$$\sum_{j=1}^3 \partial_j \hat{A}_{\text{Coul}}^j(x) = 0. \quad (4.46)$$

Restricting the $\hat{A}_{\text{Coul}}^j(x)$ to the domain D_{Coul} (compare (4.34)), which they leave invariant and on which $(\cdot | \cdot) = \langle \cdot | \cdot \rangle$, we get quantized potentials of the free electromagnetic field in a pre-Hilbert space with positive definite metric (**radiation gauge**, the special case (4.45) of the **Coulomb gauge**, characterized by (4.46)).

The price to be paid for positive definiteness of the metric is invalidity of the covariance condition (4.6) – for **every** representation $\hat{V}(a, \Lambda)$ – as well as of the locality condition (4.8). This follows from (4.45) and the structure of the 2-point function²⁴

$$\left(\Omega | \hat{A}_{\text{Coul}}^j(x) \hat{A}_{\text{Coul}}^k(y) \right) = i\zeta \left(\delta_{jk} - \frac{\partial^j \partial^k}{\Delta} \right) \Delta_0^{(+)}(x - y). \quad (4.47)$$

(4.44) shows that, for $j \in \{1, 2, 3\}$ and real-valued $\varphi \in \mathcal{S}(\mathbb{R}^4)$, $\hat{F}_{\text{GB}}^{0j}(\varphi)$ and $\hat{F}_{\text{Coul}}^{0j}(\varphi)$ are equivalent GUPTA-BLEULER observables, although not identical.

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²³Note that $p^0 \hat{c}^k(\mathbf{p}) - p^k \hat{c}^0(\mathbf{p}) = -\frac{p^k}{p^0} p_\mu \hat{a}^\mu(\mathbf{p})$ for $k \in \{1, 2, 3\}$ and that, by (4.33) and (4.26),

$\partial_\mu \hat{A}_{\text{GB}}^\mu(\varphi) D_F \subset D_{00}$.

²⁴The $\frac{\partial^j \partial^k}{\Delta}$ -term spoils commutativity.

Special Observables

Using the commutation relations (4.22), one easily checks that

$$\hat{P}_{\text{GB}}^\alpha = \int_{p^0=|\mathbf{p}|} p^\alpha (-\hat{a}^\mu(\mathbf{p})^\dagger \hat{a}_\mu(\mathbf{p})) \frac{d\mathbf{p}}{2p^0}. \quad (4.48)$$

is the generator of space-time translations:

$$i \left[\hat{A}_{\text{GB}}^\alpha, \hat{A}_{\text{GB}}^{(\pm)\mu}(x) \right]_- = \partial^\alpha \hat{A}_{\text{GB}}^{(\pm)\mu}(x). \quad (4.49)$$

Therefore, $\hat{P}_{\text{GB}}^\alpha$ is the GUPTA-BLEULER observable of linear 4-momentum specified by

$$\hat{P}_{\text{GB}}^\alpha \Omega = 0. \quad (4.50)$$

This is consistent with the interpretation of

$$\frac{-1}{2|\mathbf{p}|} \hat{a}^\mu(\mathbf{p})^\dagger \hat{a}_\mu(\mathbf{p})$$

as a GUPTA-BLEULER observable for

$$\lim_{\tilde{V}_{\mathbf{p}} \rightarrow \{\mathbf{p}\}} \frac{\text{number of physical photons with momentum } \mathbf{p}' \in \tilde{V}_{\mathbf{p}}}{\text{momentum space volume } \tilde{V}_{\mathbf{p}}}.$$

Remark: Given $\varphi \in S(\mathbb{R}^3)$, we always have

$$\int \hat{a}^\mu(\mathbf{p})^\dagger \hat{a}_\mu(\mathbf{p}) \varphi(\mathbf{p}) d\mathbf{p} \quad D \subset D \quad \text{for } D = D_{\text{F}} \text{ as well as for } D = D_{00},$$

but **not** $\hat{a}^\mu(\mathbf{p})^\dagger \hat{a}_\mu(\mathbf{p}) = 0$ on D_{00} .

Exercise 61 Show that the GUPTA-BLEULER observables $c \hat{P}_{\text{GB}}^0$ and

$$\hat{E}_{\text{GB}}(x^0) \stackrel{\text{def}}{=} \frac{\epsilon'_0 c'^2}{2} \int : \left(g_{\alpha\beta} g_{\gamma\delta} \hat{F}_{\text{AGB}}^{\alpha\gamma}(x) \hat{F}_{\text{AGB}}^{\beta\delta}(x) + 4 g_{\mu\nu} \hat{F}_{\text{AGB}}^{0\mu}(x) \hat{F}_{\text{AGB}}^{0\nu}(x) \right) : d\mathbf{x}$$

are equivalent (**not** identical) if

$$\zeta = \frac{1}{\epsilon'_0} \left(\frac{\hbar}{c'} \right)^2. \quad (4.51)$$

In the following ζ will always be assumed given by (4.51), with ϵ'_0 and c' specified by the used system of units²⁵ (see Appendix A.3 of (Lücke, edyn)).

²⁵In HEAVISIDE units: $\epsilon'_0 = c' = 1$.

Similarly to (4.49) one may show that

$$\begin{aligned}\hat{\mathbf{J}}_{\text{GB}} &= \hat{\mathbf{L}}_{\text{GB}} + \hat{\mathbf{S}}_{\text{GB}}, \quad \text{where} \\ \hat{\mathbf{L}}_{\text{GB}} &\stackrel{\text{def}}{=} \int_{p^0=|\mathbf{p}|} -\hat{a}^\mu(\mathbf{p})^\dagger \mathbf{p} \times \frac{1}{i} \nabla_{\mathbf{p}} \hat{a}_\mu(\mathbf{p}) \frac{d\mathbf{p}}{2p^0}, \\ \hat{\mathbf{S}}_{\text{GB}} &\stackrel{\text{def}}{=} i \int_{p^0=|\mathbf{p}|} \hat{\mathbf{a}}(\mathbf{p})^\dagger \times \hat{\mathbf{a}}(\mathbf{p}) \frac{d\mathbf{p}}{2p^0},\end{aligned}\tag{4.52}$$

is a generator of spatial rotations:

$$\begin{aligned}\left[\hat{\mathbf{L}}_{\text{GB}}, \hat{A}_{\text{GB}}^{(\pm)\mu}(x) \right] &= i \mathbf{x} \times \nabla_{\mathbf{x}} \hat{A}_{\text{GB}}^{(\pm)\mu}(x), \\ \left[\mathbf{e} \cdot \hat{\mathbf{S}}_{\text{GB}}, \overrightarrow{\hat{A}_{\text{GB}}^{(\pm)}}(x) \right] &= i \mathbf{e} \times \overrightarrow{\hat{A}_{\text{GB}}^{(\pm)}}(x).\end{aligned}\tag{4.53}$$

Hence $\hat{\mathbf{J}}_{\text{GB}}$ is the GUPTA-BLEULER observable of total angular momentum specified by

$$\hat{\mathbf{J}}_{\text{GB}} \Omega = 0.$$

Warning: $\hat{\mathbf{L}}_{\text{GB}}$ and $\hat{\mathbf{S}}_{\text{GB}}$ themselves are **not** GUPTA-BLEULER observables since they do not leave D_F invariant!

This is consistent with the interpretation of

$$\frac{1}{2|\mathbf{p}|} \left(-\hat{a}^\mu(\mathbf{p})^\dagger \mathbf{p} \times \frac{1}{i} \nabla_{\mathbf{p}} \hat{a}_\mu(\mathbf{p}) + i \hat{\mathbf{a}}(\mathbf{p})^\dagger \times \hat{\mathbf{a}}(\mathbf{p}) \right)$$

as GUPTA-BLEULER observable of

$$\lim_{\tilde{V}_{\mathbf{p}} \rightarrow \{\mathbf{p}\}} \frac{\text{total angular momentum of all photons with momentum } \mathbf{p}' \in \tilde{V}_{\mathbf{p}}}{\text{momentum space volume } \tilde{V}_{\mathbf{p}}}.$$

Hence, the so-called *helicity operator*

$$\hat{\lambda}_{\text{GB}} \stackrel{\text{def}}{=} i \int_{p^0=|\mathbf{p}|} \frac{\mathbf{p}}{|\mathbf{p}|} \cdot (\hat{\mathbf{a}}(\mathbf{p})^\dagger \times \hat{\mathbf{a}}(\mathbf{p})) \frac{d\mathbf{p}}{2p^0}\tag{4.54}$$

is a GUPTA-BLEULER observable for the component of angular momentum along three-momentum.

n-photon states are called those represented by the elements of D_A which are of the form

$$A_{\text{GB}} = \{0, \dots, 0, A_n, 0, \dots\}$$

(compare 4.1.3). The GUPTA-BLEULER observable for the number of **physical** photons is

$$\hat{N}_{\text{GB}} \stackrel{\text{def}}{=} - \int_{p^0=|\mathbf{p}|} \hat{a}^\mu(\mathbf{p})^\dagger \hat{a}_\mu(\mathbf{p}) \frac{d\mathbf{p}}{2p^0}\tag{4.55}$$

(recall the comment to (4.48)).

Remark: The expectation values for the field strengths in physical n -photon states all vanish!

Every 1-photon state $A_{\text{GB}} = \{0, A_1, 0, 0, \dots\}$ has a unique decomposition

$$A_{\text{GB}} = A_{\text{GB}}^{\text{tr}} + A_{\text{GB}}^{\text{lo}} + A_{\text{GB}}^{\text{ti}}$$

with:

$$\begin{aligned} A_{\text{GB}}^{\text{tr}} &\in D_{\text{Coul}}, & \text{i.e. } p^j (A_{\text{GB}}^{\text{tr}})^j(\mathbf{p}) &= (A_{\text{GB}}^{\text{tr}})^0(\mathbf{p}) = 0, \\ A_{\text{GB}}^{\text{lo}} &\in D_{00}, & \text{i.e. } (A_{\text{GB}}^{\text{lo}})^j(\mathbf{p}) &= \frac{p^j}{|\mathbf{p}|} (A_{\text{GB}}^{\text{lo}})^0(\mathbf{p}), \\ (A_{\text{GB}}^{\text{ti}})^j(\mathbf{p}) &= 0 & \text{for } j = 1, 2, 3. \end{aligned}$$

One says:

$A_{\text{GB}}^{\text{tr}}$ describes a **transverse photon**, $A_{\text{GB}}^{\text{lo}}$ a **longitudinal photon**, and $A_{\text{GB}}^{\text{ti}}$ a **time-like photon**.

Of course, the 1-photon state vector A_{GB} corresponds to a **physical state** only if $A_{\text{GB}}^{\text{ti}} = 0$. In this case A_{GB} and $A_{\text{GB}}^{\text{tr}}$ correspond to the same physical state.

The commutation relations

$$i \left[\hat{a}^0(\mathbf{p})^\dagger, \hat{\lambda}_{\text{GB}} \right]_- = 0, \quad i \left[\mathbf{e} \cdot \hat{\mathbf{a}}(\mathbf{p})^\dagger, \hat{\lambda}_{\text{GB}} \right]_- = \frac{\mathbf{p}}{|\mathbf{p}|} \cdot (\mathbf{e} \times \hat{\mathbf{a}}(\mathbf{p})^\dagger) \quad (4.56)$$

show²⁶ that the longitudinal (as well as the time-like) 1-photon state vectors are eigenvectors of $\hat{\lambda}_{\text{GB}}$ with eigenvalue 0. Moreover, we see that the transversal 1-photon state vectors of the form

$$\begin{aligned} &\int_{p^0=|\mathbf{p}|} \varphi(\mathbf{p}) (\mathbf{e}_1(\mathbf{p}) \cdot \hat{\mathbf{a}}(\mathbf{p})^\dagger - i\sigma \mathbf{e}_2(\mathbf{p}) \cdot \hat{\mathbf{a}}(\mathbf{p})^\dagger) \frac{d\mathbf{p}}{2p^0} \Omega, \\ \text{where: } &\left\{ \mathbf{e}_1(\mathbf{p}), \mathbf{e}_2(\mathbf{p}), \frac{\mathbf{p}}{|\mathbf{p}|} \right\} \text{ right-handed orthonormal basis of } \mathbb{R}^3 \forall \mathbf{p} \neq 0, \end{aligned} \quad (4.57)$$

with $\sigma \in \{+1, -1\}$ are eigenvectors of $\hat{\lambda}_{\text{GB}}$ with Eigenvalue σ , hence correspond to physical 1-photon states with **helicity** σ . Obviously, every transversal 1-photon state vector may be written as a linear combination of vectors of the form (4.57).

4.2 The Quantized Free DIRAC Field

²⁶Note that $\hat{\lambda}_{\text{GB}} \Omega = 0$.

4.2.1 LORENTZ Transformations Characterized via Complex 2×2 -Matrices

Every selfadjoint complex 2×2 -matrix \tilde{X} may be written in the form

$$\tilde{X} = x^\mu \tau_\mu = \begin{pmatrix} x^0 - x^3 & -x^1 + ix^2 \\ -x^1 - ix^2 & x^0 + x^3 \end{pmatrix} \quad (4.58)$$

with suitable $x \in \mathbb{R}^4$, where

$$\tau_0 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tau_1 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tau_2 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \tau_3 \stackrel{\text{def}}{=} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.59)$$

Exercise 62 Show that²⁷

$$A^2 = \text{Tr}(A) A - \det(A) \mathbb{1}_2$$

holds for all complex 2×2 -matrices A .

Since²⁸

$$\text{Tr}(\tau_\mu \tau_\nu) = 2 \delta_{\mu\nu} \quad (4.60)$$

the coefficients in (4.58) are

$$x^\mu = \frac{1}{2} \text{Tr}(\tau_\mu \tilde{X}). \quad (4.61)$$

Moreover,²⁹

$$\det(A \tilde{X} A^*) = \det(\tilde{X}) = x^\mu x_\mu \quad \forall A \in \text{SL}(2, \mathbb{C}) \quad (4.62)$$

implies

$$x'^\mu x'_\mu = x^\mu x_\mu \quad \text{if } \tilde{X}' = A \tilde{X} A^*.$$

Hence, by (4.61), for every $A \in \text{SL}(2, \mathbb{C})$

$$x^\mu \mapsto x'^\mu = (\Lambda_A)^\mu_\nu x^\nu, \quad (4.63)$$

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²⁷This is a special case of a well-known theorem by Caley stating for arbitrary $n \in \mathbb{N}$ that

$$\left(c_M(z) \stackrel{\text{def}}{=} \det(M - \lambda \mathbb{1}_n) \quad \text{for } z \in \mathbb{C} \right) \implies c_M(M) = 0$$

holds for all $n \times n$ -matrices M .

²⁸This is a simple consequence of:

$$\frac{1}{2} [\tau^\mu, \tau^\nu]_+ = \begin{cases} \delta_{\mu\nu} & \text{if } \mu, \nu \in \{1, 2, 3\}, \\ \tau^\mu & \text{if } \nu = 0 \end{cases}$$

(compare (4.68)).

²⁹As usual, $\text{SL}(2, \mathbb{C})$ denotes the group of all complex 2×2 -matrices X with $\det X = 1$.

where

$$\boxed{(\Lambda_A)^\mu{}_\nu \stackrel{\text{def}}{=} \frac{1}{2} \text{Tr}(\tau_\mu A \tau_\nu A^*)}, \quad (4.64)$$

is a restricted Lorentz transformation,³⁰ $\Lambda_A \in L_+^\uparrow$, depending continuously on A . Note that $A \mapsto \Lambda_A$ is a representation of $\text{SL}(2, \mathbb{C})$:

$$\Lambda_A \Lambda_B = \Lambda_{AB} \quad \forall A, B \in \text{SL}(2, \mathbb{C}). \quad (4.65)$$

The subgroup of all unitary elements in $\text{SL}(2, \mathbb{C})$ is³¹

$$\text{SU}(2) = \{U_\varphi : \varphi \in \mathbb{R}^3, |\varphi| \leq 2\pi\}, \quad U_\varphi \stackrel{\text{def}}{=} \exp\left(-i \boldsymbol{\tau} \cdot \frac{\boldsymbol{\varphi}}{2}\right), \quad (4.66)$$

where the components of $\boldsymbol{\tau}$ are the PAULI *matrices*

$$\tau^j \stackrel{\text{def}}{=} -\tau_j \quad \text{for } j \in \{1, 2, 3\}, \quad (4.67)$$

fulfilling³²

$$\tau^j \tau^k = \delta_{jk} + i \sum_{l=1}^3 \epsilon_{jkl} \tau^l \quad (4.68)$$

and hence

$$(\boldsymbol{\tau} \cdot \boldsymbol{\varphi})^2 = |\boldsymbol{\varphi}|^2 \quad \forall \boldsymbol{\varphi} \in \mathbb{R}^3.$$

Exercise 63 Prove

$$U_\varphi = \mathbb{1}_2 \cos \frac{|\boldsymbol{\varphi}|}{2} - i \frac{\boldsymbol{\tau} \cdot \boldsymbol{\varphi}}{|\boldsymbol{\varphi}|} \sin \frac{|\boldsymbol{\varphi}|}{2},$$

(4.66), and:³³

$$U_{\boldsymbol{\varphi}'} = U_\varphi \iff (\boldsymbol{\varphi}' = \boldsymbol{\varphi} \vee |\boldsymbol{\varphi}'| = |\boldsymbol{\varphi}| = 2\pi).$$

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³⁰Here we identify the active Lorentz transformations with their matrix realizations w.r.t. some fixed orthonormal inertial system. The $A \in \text{SL}(2, \mathbb{C})$ with $\Lambda_A = \Lambda$ for given $\Lambda \in L_+^\uparrow$ are determined in (Macfarlane, 1962). That (4.64) defines **restricted** Lorentz transformations follows from the fact that for every $A \in \text{SL}(2, \mathbb{C})$ there is a continuous path connecting Λ_A with $\Lambda_{\tau_0} = \mathbb{1}_4$.

³¹Note that $\det(e^{iJ}) = e^{i \text{Tr}(J)}$ for $J = J^*$, and: $\text{Tr}(J) = 0 \xrightarrow{(4.67)} J = -\boldsymbol{\tau} \cdot \frac{\boldsymbol{\varphi}}{2}$.

³²As usual, we define

$$\epsilon_{jkl} \stackrel{\text{def}}{=} \begin{cases} +1 & \text{if } (j, k, l) \text{ is an even permutation of } (1, 2, 3), \\ -1 & \text{if } (j, k, l) \text{ is an odd permutation of } (1, 2, 3), \\ 0 & \text{else.} \end{cases}$$

Hence

$$\{x^0 + x^1(i\tau^1) + x^2(i\tau^2) + x^3(i\tau^3) : x \in \mathbb{R}^4\},$$

considered as algebra over \mathbb{R}^1 , is isomorphic to the algebra of *quaternions*, generated by $\hat{i} \stackrel{\text{def}}{=} i\tau^1$ and $\hat{j} \stackrel{\text{def}}{=} i\tau^2$ (as *Clifford algebra*; see (Choquet-Bruhat et al., 1978, S. 63/64)).

³³This shows that $\text{SU}(2)$ (w.r.t. its natural topology) – contrary to the rotation group – is *simply connected*.

Now we may easily show (recall Exercise 63), that for the unitary elements

$$U_{\boldsymbol{\varphi}} \in \text{SU}(2) \subset \text{SL}(2, \mathbb{C})$$

the associated transformations $\Lambda_{U_{\boldsymbol{\varphi}}}$ are spatial rotations:³⁴

$$\Lambda_{U_{\boldsymbol{\varphi}}} = \exp \left(\frac{1}{2} \sum_{j,k,l=1}^3 \epsilon_{jkl} \mathbb{T}^{jk} \varphi^l \right), \quad \text{where:}$$

$$\mathbb{T}^{23} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \frac{d}{d\varphi} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cos \varphi & -\sin \varphi \\ 0 & 0 & \sin \varphi & \cos \varphi \end{pmatrix} \Big|_{\varphi=0}, \quad (4.69)$$

$$\mathbb{T}^{13} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbb{T}^{12} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbb{T}^{jk} \stackrel{\text{def}}{=} -\mathbb{T}^{kj} \quad \text{for } j > k.$$

Obviously, $\Lambda_{U_{\boldsymbol{\varphi}}}$ is the matrix (w.r.t. the right-handed basis in which $\boldsymbol{\varphi} = (\varphi^1, \varphi^2, \varphi^3)$) of a right-handed rotation, by the angle $|\boldsymbol{\varphi}|$, around an axis oriented along $\boldsymbol{\varphi}$. Therefore:

$$\Lambda_{U_{\boldsymbol{\varphi}}} = \Lambda_{U_{\boldsymbol{\varphi}'}} \iff U_{\boldsymbol{\varphi}'} \in \{+U_{\boldsymbol{\varphi}}, -U_{\boldsymbol{\varphi}}\}.$$

Exercise 64 Prove that every **positive** hermitian element of $\text{SL}(2, \mathbb{C})$ is of the form

$$\begin{aligned} H_{\mathbf{v}} \stackrel{\text{def}}{=} \exp \left(-\frac{\chi_{\mathbf{v}}}{2} \boldsymbol{\tau} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} \right) &= \cosh \left(\frac{\chi_{\mathbf{v}}}{2} \right) \hat{1} - \sinh \left(\frac{\chi_{\mathbf{v}}}{2} \right) \boldsymbol{\tau} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \sqrt{\frac{\mathbf{v}^0 + 1}{2}} \left(\tau^0 - \boldsymbol{\tau} \cdot \frac{\mathbf{v}}{1 + 1/\mathbf{v}^0} \right) \end{aligned} \quad (4.70)$$

with suitable $\mathbf{v} \in \mathbb{R}^3$, $|\mathbf{v}| < 1$, where:³⁵

$$\chi_{\mathbf{v}} \stackrel{\text{def}}{=} \tanh^{-1} |\mathbf{v}| \geq 0, \quad \mathbf{v}^0 \stackrel{\text{def}}{=} \frac{1}{\sqrt{1 - |\mathbf{v}|^2}}.$$

Moreover, show for arbitrary $\mathbf{v}, \mathbf{v}' \in \mathbb{R}^3$ with $|\mathbf{v}|, |\mathbf{v}'| < 1$ that

$$H_{\mathbf{v}'} = H_{\mathbf{v}} \iff \mathbf{v}' = \mathbf{v}.$$

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³⁴It is sufficient to check the generators. The mapping $U_{\boldsymbol{\varphi}} \mapsto \Lambda_{U_{\boldsymbol{\varphi}}}$ has all the properties of a **covering mapping** (Pontrjagin, 1958, Def. 45). Therefore (recall Footnote 33), $\text{SU}(2)$ is the **universal covering group** of the rotation group.

³⁵Note that $\tanh \frac{\chi_{\mathbf{v}}}{2} = \frac{\tanh \chi_{\mathbf{v}}}{1 + \sqrt{1 - \tanh^2 \chi_{\mathbf{v}}}} = \frac{|\mathbf{v}|}{1 + 1/\mathbf{v}^0}$, $\cosh = \frac{1}{\sqrt{1 - \tanh^2}}$.

Similarly (recall Exercise 64), for positive hermitian $A \in \text{SL}(2, \mathbb{C})$ the Λ_A correspond to Lorentz boosts:

$$\Lambda_{H_{\mathbf{v}}} = \exp \left(\chi_{\mathbf{v}} \sum_{j=1}^3 \mathbb{T}^{0j} \frac{v^j}{|\mathbf{v}|} \right), \quad \text{where:}$$

$$\mathbb{T}^{01} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{d}{d\chi} \begin{pmatrix} \cosh \chi & \sinh \chi & 0 & 0 \\ \sinh \chi & \cosh \chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Big|_{\chi=0}, \quad (4.71)$$

$$\mathbb{T}^{02} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{T}^{03} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Exercise 65 Prove (4.69) and (4.71).

The so-called *polar decomposition* (for invertible³⁶ A)

$$A = \underbrace{A \sqrt{A^{-1} A^{*-1}}}_{\text{unitary}} \underbrace{\sqrt{A^* A}}_{\text{pos. herm.}} \quad (4.72)$$

together with (4.69) and (4.71) shows:³⁷

$$\{\Lambda_A : A \in \text{SL}(2, \mathbb{C})\} = \{\Lambda_{A\varphi} \Lambda_{H_{\mathbf{v}}} : \varphi, \mathbf{v} \in \mathbb{R}^3, |\mathbf{v}| < 1\} = L_+^\uparrow \quad (4.73)$$

(for the last equality see, e.g., equations (2.39) and (2.37) in (Lücke, rel)).

4.2.2 Relativistic Covariance in General

Consider any relativistic quantum theory with state space \mathcal{H} . Even if the relativistic symmetries are realized as Wigner symmetries, there is no reason, why these should correspond to a true representation of \mathcal{P}_+^\uparrow . In this case, by Wigner's theorem (Theorem 1.2.1), one may choose for every (a, Λ) a unitary operator $\hat{U}(a, \Lambda)$ such that³⁸

$$\left\{ \lambda \hat{U}(a_1, \Lambda_1) \hat{U}(a_2, \Lambda_2) \Psi : \lambda \in \mathbb{C} \right\} = \left\{ \lambda \hat{U} \left((a_1, \Lambda_1) \circ (a_2, \Lambda_2) \right) \Psi : \lambda \in \mathbb{C} \right\}$$

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³⁶For singular A the unitary operator $A \sqrt{A^{-1} A^{*-1}}$ has to be replaced by a suitable isometric operator (see, e.g. (Bratteli and Robinson, 1979, p. 39).)

³⁷Note that $\det(A) = 1 \implies \det(\sqrt{A^* A}) = 1$.

³⁸Here $(a_1, \Lambda_1) \circ (a_2, \Lambda_2) \stackrel{\text{def}}{=} (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2)$ is the group operation of \mathcal{P}_+^\uparrow .

holds for every pair of Poincaré transformations $(a_1, \Lambda_1), (a_2, \Lambda_2) \in \mathcal{P}_+^\uparrow$ and all $\Psi \in \mathcal{H}$, since the action of $\hat{U}(a, \Lambda)$ is to be interpreted in the sense of (2.6) (with \hat{f} replaced by an arbitrary element of \mathcal{H}). But this only implies existence of a phase function $\varphi((a_1, \Lambda_1), (a_2, \Lambda_2)) \in \mathbb{R}$ with

$$\hat{U}((a_1, \Lambda_1) \circ (a_2, \Lambda_2)) = e^{i\varphi((a_1, \Lambda_1), (a_2, \Lambda_2))} \hat{U}(a_1, \Lambda_1) \hat{U}(a_2, \Lambda_2). \quad (4.74)$$

Exercise 66 Show that the phase function φ in (4.74) is a **2-cocycle** w.r.t. the trivial representation $\pi(a, \Lambda) = 1$ of \mathcal{P}_+^\uparrow in \mathbb{R} , i.e. it fulfills the condition $\delta^{(2)}\varphi = 0$, where $\delta^{(n)}$ denotes the **coboundary operator** defined by

$$\begin{aligned} (\delta^{(n)}f)(g_1, \dots, g_{n+1}) &\stackrel{\text{def}}{=} \pi(g_1)f(g_2, \dots, g_{n+1}) + \\ &+ \sum_{\nu=1}^n (-1)^\nu f(g_1, \dots, g_\nu \circ g_{\nu+1}, \dots, g_{n+1}) + (-1)^{n+1} f(g_1, \dots, g_n) \end{aligned}$$

((Van Est, 1953, Eq. 25)). Moreover, show that φ may be eliminated by suitable change of phase ($\hat{U}(g) \rightarrow e^{ih(g)}\hat{U}(g)$) iff φ is a **1-coboundary**, i.e. of the form $\varphi = \delta^{(1)}h$.

In this case, we still have

$$\left\{ \lambda \hat{U}(a_1, \Lambda_1) \hat{U}(a_2, \Lambda_2) : \lambda \in \mathbb{C} \right\} = \left\{ \lambda \hat{U}((a_1, \Lambda_1) \circ (a_2, \Lambda_2)) : \lambda \in \mathbb{C} \right\}$$

for every pair of Poincaré transformations $(a_1, \Lambda_1), (a_2, \Lambda_2) \in \mathcal{P}_+^\uparrow$, i.e.

$$(a, \Lambda) \mapsto \mathcal{U}(a, \Lambda) \stackrel{\text{def}}{=} \left\{ \lambda \hat{U}(a, \Lambda) : \lambda \in \mathbb{C} \right\}$$

is a unitary **ray representation** of \mathcal{P}_+^\uparrow .

Fortunately, according to Bargmann (Bargmann, 1954) the following holds:³⁹

Theorem 4.2.1 *Let $\mathcal{U}(a, \Lambda)$ be a continuous⁴⁰ ray representation of \mathcal{P}_+^\uparrow in \mathcal{H} . then there is a continuous unitary representation $\hat{U}(a, A)$ of⁴¹ $iSL(2, \mathbb{C})$ in \mathcal{H} with:*

$$\mathcal{U}(a, \Lambda) = \left\{ \lambda \hat{U}(a, A) : \lambda \in \mathbb{C} \right\} \quad \forall (a, A) \in iSL(2, \mathbb{C}).$$

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³⁹See also (Varadarajan, 2007, Sect. VIII.5) and, for 1 + 2 dimensions, (Grigore, 1993). Ray representations of the Galilei group may always be considered as true representations of some central extension of the Galilei group (Levy-Leblond, 1963).

⁴⁰Here continuity is to be understood in the sense of Condition (iii) of Definition 1.1.4.

⁴¹As usual, $iSL(2, \mathbb{C})$ denotes the group

$$iSL(2, \mathbb{C}) = \{(a, A) : A \in SL(2, \mathbb{C}), a \in \mathbb{R}^4\}$$

In this sense, $\text{iSL}(2, \mathbb{C})$ is more fundamental than \mathcal{P}_+^\uparrow :

If the symmetries which express – in the sense of special relativity – the equivalence of all inertial systems are Wigner symmetries then they are given by some⁴² continuous unitary representation of $\text{iSL}(2, \mathbb{C})$.

4.2.3 DIRAC Particles

Massive Spin- $\frac{1}{2}$ Representations of $\text{iSL}(2, \mathbb{C})$

The simplest non-scalar unitary representations⁴³ of $\text{iSL}(2, \mathbb{C})$ are of the form⁴⁴

$$\left(\hat{U}(a, A)\chi\right)(\mathbf{p}) \stackrel{\text{def}}{=} e^{(ip^\mu a_\mu)} A \chi\left(\overrightarrow{\Lambda_{A^{-1}} p}\right)_{|p^0=\omega_{\mathbf{p}}}; \quad \omega_{\mathbf{p}} \stackrel{\text{def}}{=} \sqrt{m^2 + \mathbf{p}^2}, \quad (4.75)$$

where m is some fixed **positive** mass and the representation space is the set of all \mathbb{C}^2 -valued wave functions $\chi(\mathbf{p})$ with finite HILBERT space norm

$$\|\chi\| \stackrel{\text{def}}{=} \sqrt{\int_{p^0=\omega_{\mathbf{p}}} \chi(\mathbf{p})^* \frac{p^\mu \sigma_\mu}{m} \chi(\mathbf{p}) \frac{d\mathbf{p}}{2p^0}}, \quad \sigma_\mu \stackrel{\text{def}}{=} \tau^\mu. \quad (4.76)$$

Since

$$A^* A \stackrel{\text{i.a.}}{\neq} \mathbb{1}_2,$$

the term $p^\mu \sigma_\mu$ fulfilling

$$A^* p^\mu \sigma_\mu A = (\Lambda_{A^{-1}} p)^\mu \sigma_\mu \quad (4.77)$$

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with multiplication

$$(a_1, A_1) \circ (a_2, A_2) = (a_1 + \Lambda_{A_1} a_2, A_1 A_2).$$

Since $\text{SU}(2)$ is simply connected (recall Footnote 33), (4.71) shows that the same is true for $\text{iSL}(2, \mathbb{C})$.

⁴²Of course, not all continuous unitary representations of $\text{iSL}(2, \mathbb{C})$ are physically relevant.

⁴³See, e.g. (Streater and Wightman, 1989, p. 15) for a characterization of all irreducible representations of $\text{SL}(2, \mathbb{C})$.

⁴⁴Note that $\Lambda_{A^{-1}} \stackrel{(4.65)}{=} (\Lambda_A)^{-1}$.

is needed to make the representation (4.75) unitary.

Sketch of proof for (4.77): With $\sigma^\mu \stackrel{\text{def}}{=} g^{\mu\nu} \sigma_\nu = \tau_\mu$ we have

$$\begin{aligned}
 \text{Tr}(\tau_\lambda (\Lambda_{A^{-1}} p)^\mu \sigma_\mu) &= \text{Tr}(\tau_\lambda p_\mu (\Lambda_A)^\mu{}_\nu \sigma^\nu) \\
 &\stackrel{(4.60)}{=} 2p_\mu (\Lambda_A)^\mu{}_\lambda \\
 &\stackrel{(4.64)}{=} \text{Tr}(p_\mu \sigma^\mu A \tau_\lambda A^*) \\
 &= \text{Tr}(\tau_\lambda A^* p^\mu \sigma_\mu A) \quad \text{for } \lambda = 0, 1, 2, 3.
 \end{aligned}$$

Together with (4.60) this implies (4.77). ■

Exercise 67 Show that⁴⁵ $p^\mu p_\mu > 0 \implies p^\mu \sigma_\mu \geq 0$ and that the representation (4.75) of $\text{iSL}(2, \mathbb{C})$ is irreducible.

According to the action of $\Lambda_{U_{\varphi\mathbf{e}}}$, the components of the the operator $\hat{\mathbf{J}}$ defined by (4.75) and

$$\mathbf{e} \cdot \hat{\mathbf{J}} \stackrel{\text{def}}{=} i \frac{d}{d\varphi} \hat{U}(0, U_{\varphi\mathbf{e}})|_{\varphi=0} = \mathbf{e} \cdot \left(\underbrace{i\mathbf{p} \times \nabla_{\mathbf{p}}}_{\stackrel{\text{def}}{=} \hat{\mathbf{L}}} + \underbrace{\frac{1}{2}\boldsymbol{\tau}}_{\stackrel{\text{def}}{=} \hat{\mathbf{S}}} \right) \quad (4.78)$$

are interpreted as observables of total angular momentum.

Exercise 68 For $\hat{\mathbf{S}}$ given by (4.78) prove

$$A_\varphi(\mathbf{e} \cdot \hat{\mathbf{S}}) A_\varphi^{-1} = (\hat{D}_\varphi \mathbf{e}) \cdot \hat{\mathbf{S}} \quad \forall \mathbf{e} \in \mathbb{R}^3, \quad (4.79)$$

where \hat{D}_φ denotes right-handed rotation by the angle $|\varphi|$ around an axis oriented along φ , and use this to determine the eigenstates of $\mathbf{e} \cdot \hat{\mathbf{S}}$ for arbitrary $\mathbf{e} \in \mathbb{R}^3$.

(4.78) becomes especially simple for $\mathbf{e} = \frac{\mathbf{p}}{|\mathbf{p}|}$:

$$\hat{h} \stackrel{\text{def}}{=} \frac{\mathbf{p}}{|\mathbf{p}|} \cdot \hat{\mathbf{J}} = \frac{\mathbf{p}}{|\mathbf{p}|} \cdot \hat{\mathbf{S}} \quad \textit{helicity operator}. \quad (4.80)$$

Obviously,

$$\hat{h} \chi_\pm(\mathbf{p}) = \pm \frac{1}{2} \chi_\pm(\mathbf{p}) \quad (4.81)$$

⁴⁵Check $\det(p^\mu \sigma_\mu)$ and $\text{Tr}(p^\mu \sigma_\mu)$.

holds for

$$\begin{aligned}
\chi_+ \left(|\mathbf{p}| (\cos \vartheta \cos \varphi \mathbf{e}_1 + \sin \vartheta \sin \varphi \mathbf{e}_2 + \cos \vartheta \mathbf{e}_3) \right) &\stackrel{\text{def}}{=} A_{\varphi \mathbf{e}_3} A_{\vartheta \mathbf{e}_2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} +e^{-i\frac{\varphi}{2}} \cos \frac{\vartheta}{2} \\ +e^{+i\frac{\varphi}{2}} \sin \frac{\vartheta}{2} \end{pmatrix}, \\
\chi_- \left(|\mathbf{p}| (\cos \vartheta \cos \varphi \mathbf{e}_1 + \sin \vartheta \sin \varphi \mathbf{e}_2 + \cos \vartheta \mathbf{e}_3) \right) &\stackrel{\text{def}}{=} A_{\varphi \mathbf{e}_3} A_{\vartheta \mathbf{e}_2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -e^{-i\frac{\varphi}{2}} \sin \frac{\vartheta}{2} \\ +e^{+i\frac{\varphi}{2}} \cos \frac{\vartheta}{2} \end{pmatrix}
\end{aligned} \tag{4.82}$$

(compare Exercise 68).

The helicity survives in the limit $\frac{m}{|\mathbf{p}|} \rightarrow 0$ (to be discussed at the end of this section) in which, however, the representation of $\text{iSL}(2, \mathbb{C})$ becomes reducible (neutrinos!).

The wave functions may be written in the form

$$\chi(\mathbf{p}) = \sum_{\sigma=\pm} b_{\sigma}(\mathbf{p}) \chi_{\sigma}(\mathbf{p}), \tag{4.83}$$

where

$$\chi_+(\mathbf{p}) \stackrel{\text{def}}{=} H_{\frac{\mathbf{p}}{\omega_{\mathbf{p}}}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_-(\mathbf{p}) \stackrel{\text{def}}{=} H_{\frac{\mathbf{p}}{\omega_{\mathbf{p}}}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.84}$$

Then, by (4.77), the norm (4.76) becomes

$$\|\chi\| = \sqrt{\sum_{\sigma=\pm} \int |b_{\sigma}(\mathbf{p})|^2 \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}}. \tag{4.85}$$

Moreover,

$$\begin{aligned}
(4.75), (4.83) &\implies \left(\hat{U}(a, A) \chi \right) (\mathbf{p}) = \sum_{\sigma=\pm} b'_{\sigma}(\mathbf{p}) \chi_{\sigma}(\mathbf{p}), \text{ where:} \\
\begin{pmatrix} b'_+(\mathbf{p}) \\ b'_-(\mathbf{p}) \end{pmatrix} &= e^{ip^{\mu} a_{\mu}} \left(H_{\frac{-\mathbf{p}}{\omega_{\mathbf{p}}}} A H_{\frac{\mathbf{p}'}{\omega_{\mathbf{p}'}}} \right) \begin{pmatrix} b_+(\mathbf{p}') \\ b_-(\mathbf{p}') \end{pmatrix}, \quad p' \stackrel{\text{def}}{=} (\Lambda_A^{-1} p)_{|_{p^0=\omega_{\mathbf{p}}}}.
\end{aligned} \tag{4.86}$$

Proof of (4.86):

$$\begin{aligned}
A \chi(\mathbf{p}') &\stackrel{(4.83)}{=} \sum_{\sigma} b_{\sigma}(\mathbf{p}') A \chi_{\sigma}(\mathbf{p}') \\
&\stackrel{(4.84)}{=} \sum_{\sigma} b_{\sigma}(\mathbf{p}') H_{\frac{\pm \mathbf{p}}{\omega_{\mathbf{p}}}} H_{\frac{-\mathbf{p}}{\omega_{\mathbf{p}}}} A H_{\frac{\mathbf{p}'}{\omega_{\mathbf{p}'}}} \chi_{\sigma}(0) \\
&= \sum_{\sigma} b_{\sigma}(\mathbf{p}') H_{\frac{\pm \mathbf{p}}{\omega_{\mathbf{p}}}} \sum_{\sigma'} \left(H_{\frac{-\mathbf{p}}{\omega_{\mathbf{p}}}} A H_{\frac{\mathbf{p}'}{\omega_{\mathbf{p}'}}} \right)_{\sigma' \sigma} \chi_{\sigma'}(0) \\
&\stackrel{(4.84)}{=} \sum_{\sigma'} \left(\sum_{\sigma} \left(H_{\frac{-\mathbf{p}}{\omega_{\mathbf{p}}}} A H_{\frac{\mathbf{p}'}{\omega_{\mathbf{p}'}}} \right)_{\sigma' \sigma} b_{\sigma}(\mathbf{p}') \right) \chi_{\sigma'}(\mathbf{p}). \quad \blacksquare
\end{aligned}$$

Exercise 69 Show for arbitrary $A \in \text{SL}(2, \mathbb{C})$ that⁴⁶

$$p' \stackrel{\text{def}}{=} (\Lambda_A^{-1} p) \Big|_{p^0=\omega_{\mathbf{p}}} \implies H_{\frac{-\mathbf{p}}{\omega_{\mathbf{p}}}} A H_{\frac{\mathbf{p}'}{\omega_{\mathbf{p}'}}} \text{ unitary,}$$

as to be expected (compare, e.g., Sect. 2.4.3 of (Lücke, rel)).

Positive Frequency Wave Functions

The transition

$$\chi(\mathbf{p}) \longrightarrow (2\pi)^{-\frac{3}{2}} \int_{p^0=\omega_{\mathbf{p}}} \chi(\mathbf{p}) e^{-ip^\mu x_\mu} \frac{d\mathbf{p}}{2p^0}$$

would give a usable configuration space version of the above representation (local transformation behavior). However, **technically** more convenient (for later inclusion of anti-particles) is another, unitarily equivalent, representation in the HILBERT space of 4-component momentum space wave functions

$$\hat{\Psi}^{(+)}(\mathbf{p}) = \sum_{\sigma=\pm} b_\sigma(\mathbf{p}) \omega_\sigma^{(+)}(\mathbf{p}) \quad (4.87)$$

with norm

$$\|\hat{\Psi}^{(+)}\| \stackrel{\text{def}}{=} \sqrt{\sum_{\sigma=\pm} \int |b_\sigma(\mathbf{p})|^2 \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}} \quad (4.88)$$

where

$$\omega_+^{(+)}(\mathbf{p}) \stackrel{\text{def}}{=} \frac{S\left(H_{\frac{\mathbf{p}}{\omega_{p^0}}}\right)}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \omega_-^{(+)}(\mathbf{p}) \stackrel{\text{def}}{=} \frac{S\left(H_{\frac{\mathbf{p}}{\omega_{p^0}}}\right)}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}; \quad (4.89)$$

namely:

$$\left(\hat{U}(a, A)\hat{\Psi}^{(+)}\right)(\mathbf{p}) \stackrel{\text{def}}{=} e^{ip^\mu a_\mu} S(A) \hat{\Psi}^{(+)}\left(\overrightarrow{\Lambda_{A^{-1}} p}\right) \Big|_{p^0=\omega_{\mathbf{p}}}, \quad (4.90)$$

where

$$S(A) \stackrel{\text{def}}{=} \begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix}. \quad (4.91)$$

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⁴⁶**Hint:** First, show that (4.70) implies

$$(H_{\mathbf{p}/p^0})^2 \Big|_{p^0=\omega_{\mathbf{p}}} = p^\mu \tau_\mu / m \quad \forall \mathbf{p} \in \mathbb{R}^3$$

and therefore

$$H_{\frac{-\mathbf{p}'}{\omega_{\mathbf{p}'}}} = \sqrt{\left(H_{\frac{-\mathbf{p}}{\omega_{\mathbf{p}}}} A\right)^{-1} \left(H_{\frac{-\mathbf{p}}{\omega_{\mathbf{p}}}} A\right)^{* -1}},$$

(see proof of (4.94)). Then recall the polar decomposition (4.72).

Exercise 70 Show that

$$(4.90), (4.87) \implies \left(\hat{U}(a, A) \hat{\Psi}^{(+)} \right) (\mathbf{p}) = \sum_{\sigma=\pm} b'_\sigma(\mathbf{p}) \omega_\sigma^{(+)}(\mathbf{p})$$

holds with $b'_\sigma(\mathbf{p})$ given by (4.86).

(4.88) may be written in the form

$$\left\| \hat{\Psi}^{(+)} \right\| = \sqrt{\int_{p^0=\omega_{\mathbf{p}}} \hat{\Psi}^{(+)}(\mathbf{p})^* \frac{p_\mu \alpha^\mu}{m} \hat{\Psi}^{(+)}(\mathbf{p}) \frac{d\mathbf{p}}{2p^0}}, \quad (4.92)$$

where

$$\alpha^\mu \stackrel{\text{def}}{=} \begin{pmatrix} \tau_\mu & 0 \\ 0 & \tau^\mu \end{pmatrix}, \quad (4.93)$$

since now the generalization

$$S(A)^* p_\mu \alpha^\mu S(A) = (\Lambda_{A^{-1}p})_\mu \alpha^\mu \quad (4.94)$$

of (4.77) holds.

Sketch of proof for (4.94):

$$\begin{aligned} \text{Tr}(\tau_\lambda (\Lambda_{A^{-1}p})^\mu \tau_\mu) &\stackrel{(4.60)}{=} 2 (\Lambda_{A^{-1}})^\lambda{}_\nu p^\nu \\ &\stackrel{(4.64)}{=} \text{Tr}(p^\nu \tau_\lambda A^{-1} \tau_\nu A^{*-1}). \end{aligned}$$

$$\stackrel{(4.61)}{\rightsquigarrow} A^{-1} p^\nu \tau_\nu A^{*-1} = (\Lambda_{A^{-1}p})^\mu \tau_\mu.$$

Together with (4.77) and (4.93) this implies (4.94). ■

With

$$\gamma^0 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma^j \stackrel{\text{def}}{=} \gamma^0 \alpha^j = \begin{pmatrix} 0 & \tau^j \\ \tau_j & 0 \end{pmatrix} \quad (4.95)$$

(4.94) becomes equivalent to

$$S(A^{-1}) \gamma^\mu S(A) = (\Lambda_A)^\mu{}_\nu \gamma^\nu, \quad (4.96)$$

thanks to

$$S(A)^* \gamma^0 = \gamma^0 S(A)^{-1}. \quad (4.97)$$

By (4.90), (4.96) is equivalent to

$$\left[\gamma^\mu p_\mu, \hat{U}(a, A) \right]_- = 0. \quad (4.98)$$

Since $\hat{U}(a, A)$ is irreducible and⁴⁷

$$\det(\gamma^\mu p_\mu) \stackrel{(4.95)}{=} \det(\alpha^\mu p_\mu) = m^4,$$

(4.98) implies:⁴⁸

$$(\gamma^\mu p_\mu - m) \hat{\Psi}^{(+)}(\mathbf{p})|_{p^0=\omega_{\mathbf{p}}} = 0. \quad (4.99)$$

Conversely, (4.99) implies that $\hat{\Psi}^{(+)}(\mathbf{p})$ is of the form (4.87).

Sketch of proof:

$$\begin{aligned} (4.99) &\iff (S(H_{\mathbf{v}})^{-1}(\gamma^\mu p_\mu - m)\hat{\psi}^{(+)}(\mathbf{p}) = 0) \\ &\stackrel{(4.96)}{\iff} \left((\gamma^0 - 1)S\left(H_{\frac{\mathbf{p}}{\omega_{\mathbf{p}}}}\right)^{-1}\hat{\psi}^{(+)}(\mathbf{p}) = 0 \right) \\ &\stackrel{(4.95)}{\iff} S\left(H_{\frac{\mathbf{p}}{\omega_{\mathbf{p}}}}\right)^{-1}\hat{\psi}^{(+)}(\mathbf{p}) = \sum_{\sigma=\pm} b_\sigma(\mathbf{p})\omega_\sigma^{(+)}(0) \\ &\stackrel{(4.89)}{\iff} \hat{\psi}^{(+)}(\mathbf{p}) = \sum_{\sigma=\pm} b_\sigma(\mathbf{p})\omega_\sigma^{(+)}(\mathbf{p}). \quad \blacksquare \end{aligned}$$

Therefore, the set of admitted configuration space wave functions

$$\Psi^{(+)}(x) \stackrel{\text{def}}{=} \sqrt{2m} (2\pi)^{-3/2} \int_{p^0=\omega_{\mathbf{p}}} \hat{\Psi}^{(+)}(\mathbf{p}) e^{-ipx} \frac{d\mathbf{p}}{2p^0} \quad (4.100)$$

coincides with the set of all normalizable **positive frequency** solutions of the DIRAC *equation*

$$(i\gamma^\mu \partial_\mu - m) \Psi(x) = 0. \quad (4.101)$$

Thanks to the factor $\sqrt{2m}$ in (4.100) we have

$$\|\hat{\Psi}^{(+)}\|^2 = \int \Psi^{(+)}(x)^* \Psi^{(+)}(x) d\mathbf{x} \quad \forall x^0 \in \mathbb{R}. \quad (4.102)$$

Proof:

$$\begin{aligned} &\int \Psi^{(+)}(x)^* \Psi^{(+)}(x) d\mathbf{x} \\ &\stackrel{(4.100)}{=} 2m \int_{p^0=\omega_{\mathbf{p}}} \hat{\Psi}^{(+)}(\mathbf{p})^* \hat{\Psi}^{(+)}(\mathbf{p}) \frac{d\mathbf{p}}{(2p^0)^2} \\ &\stackrel{(4.99)}{=} \int_{p^0=\omega_{\mathbf{p}}} \hat{\Psi}^{(+)}(\mathbf{p})^* \gamma^\mu p_\mu \hat{\Psi}^{(+)}(\mathbf{p}) \frac{d\mathbf{p}}{(2p^0)^2} + \int_{p^0=\omega_{\mathbf{p}}} \left(\gamma^\mu p_\mu \hat{\Psi}^{(+)}(\mathbf{p}) \right)^* \hat{\Psi}^{(+)}(\mathbf{p}) \frac{d\mathbf{p}}{(2p^0)^2} \\ &\stackrel{(4.109)}{=} \int_{p^0=\omega_{\mathbf{p}}} \hat{\Psi}^{(+)}(\mathbf{p})^* \gamma^0 \hat{\Psi}^{(+)}(\mathbf{p}) \frac{d\mathbf{p}}{2p^0} \\ &\stackrel{(4.99)}{=} \int_{p^0=\omega_{\mathbf{p}}} \hat{\Psi}^{(+)}(\mathbf{p})^* \frac{\gamma^0 \gamma^\mu p_\mu}{m} \hat{\Psi}^{(+)}(\mathbf{p}) \frac{d\mathbf{p}}{2p^0}. \end{aligned}$$

By (4.95) and (4.92), this implies the statement. \blacksquare

⁴⁷**Warning:** The matrix $\gamma^\mu p_\mu$ is **not** selfadjoint (compare (4.109)).

⁴⁸The correct sign may be easily determined by checking the special case $\mathbf{p} = 0$.

Exercise 71 Show that

$$S(U_\varphi) = \exp\left(-\frac{i}{4} \sum_{j,k,l=1}^3 \epsilon_{jkl} \gamma^j \gamma^k \varphi^l\right), \quad S\left(H_{\frac{\mathbf{p}}{\omega_{\mathbf{p}}}}\right) = \sqrt{\frac{\omega_{\mathbf{p}} + m}{2m}} \gamma^0 \left(\gamma^0 + \frac{\boldsymbol{\gamma} \cdot \mathbf{p}}{\omega_{\mathbf{p}} + m}\right),$$

and

$$\gamma^1 \gamma^3 \overline{S(A)} = S(A^{*-1}) \gamma^1 \gamma^3 \quad \forall A \in \text{SL}(2, \mathbb{C}),$$

where \overline{S} denotes the matrix resulting from substituting the entries of the matrix S by their complex conjugates (i.e. $\overline{S} = S^{*\text{T}}$).

Discrete Symmetries

The so-called **parity operator** \mathfrak{P} , describing total spatial reflection of the state, is fixed – up to some irrelevant constant phase factor – by⁴⁹

$$\mathfrak{P}^{-1} \hat{U}(a, A) \mathfrak{P} = \hat{U}(\mathbb{P}a, A^{*-1}), \quad \mathbb{P}x \stackrel{\text{def}}{=} (x^0, -\mathbf{x}), \quad (4.103)$$

and the requirement of unitarity (positive energy⁵⁰). Note that

$$\mathbb{P} \Lambda_A \mathbb{P} = \mathbb{T} \Lambda_A \mathbb{T} = \Lambda_{A^{*-1}} \quad \forall A \in \text{SL}(2, \mathbb{C}),$$

where

$$\mathbb{T}x \stackrel{\text{def}}{=} (-x^0, \mathbf{x}).$$

The natural choice is $\mathfrak{P}^2 = \hat{1}$, i.e.:

$$(\mathfrak{P}\Psi^{(+)}) (x) = \gamma^0 \Psi^{(+)}(\mathbb{P}x), \quad (4.104)$$

by (4.97).

Exercise 72 Show that

$$\gamma^0 \omega_\sigma^{(+)}(-\mathbf{p}) = \omega_\sigma^{(+)}(\mathbf{p})$$

and, therefore,

$$\begin{aligned} \Psi^{(+)}(x) &= \sqrt{2m} (2\pi)^{-3/2} \int_{p^0=\omega_{\mathbf{p}}} \sum_{\sigma=\pm} b_\sigma(\mathbf{p}) \omega_\sigma^{(+)}(\mathbf{p}) e^{-ipx} \frac{d\mathbf{p}}{2p^0} \\ \Rightarrow (\mathfrak{P}\Psi^{(+)}) (x) &= \sqrt{2m} (2\pi)^{-3/2} \int_{p^0=\omega_{\mathbf{p}}} \sum_{\sigma=\pm} b_\sigma(-\mathbf{p}) \omega_\sigma^{(+)}(\mathbf{p}) e^{-ipx} \frac{d\mathbf{p}}{2p^0}. \end{aligned}$$

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⁴⁹For the necessity of both conditions see, e.g., (Martin and Spearman, 1970, Chapter 5 §1).

⁵⁰By (4.103), anti-unitarity of \mathfrak{P} would imply $\mathfrak{P}^{-1} \hat{P}^0 \mathfrak{P} = -\hat{P}^0$ and thus $\langle \mathfrak{P}\Psi | \hat{P}^0 \mathfrak{P}\Psi \rangle = \overline{\langle \Psi | \mathfrak{P}^{-1} \hat{P}^0 \mathfrak{P}\Psi \rangle} = -\langle \Psi | \hat{P}^0 \Psi \rangle$ for the Hamiltonian $\hat{P}^0 = -i \frac{d}{dx^0} \hat{U}((x^0, 0, 0, 0), \mathbb{1}_4)$.

Similarly the time reversal operator \mathfrak{T} is fixed – up to some irrelevant constant phase factor – by

$$\mathfrak{T}^{-1} \hat{U}(a, A) \mathfrak{T} = \hat{U}(\mathbb{T}a, A^{*-1})$$

and the condition of **anti**-unitarity.⁵¹ The usual choice is:⁵²

$$(\mathfrak{T} \Psi^{(+)})(x) = i\gamma^1 \gamma^3 \overline{\Psi^{(+)}(\mathbb{T}x)} \quad (4.105)$$

(recall Exercise 71).

Exercise 73 Show that

$$\gamma^1 \gamma^3 \overline{\omega_{\sigma}^{(\sigma')}(+\mathbf{p})} = -\sigma \sigma' \omega_{-\sigma}^{(+)}(-\mathbf{p})$$

and, therefore,

$$\begin{aligned} \Psi^{(+)}(x) &= \sqrt{2m} (2\pi)^{-3/2} \int_{p^0=\omega_{\mathbf{p}}} \sum_{\sigma=\pm} b_{\sigma}(\mathbf{p}) \omega_{\sigma}^{(+)}(\mathbf{p}) e^{-ipx} \frac{d\mathbf{p}}{2p^0} \\ \implies (\mathfrak{T} \Psi^{(+)})(x) &= \sqrt{2m} (2\pi)^{-3/2} \int_{p^0=\omega_{\mathbf{p}}} \sum_{\sigma=\pm} i\sigma \overline{b_{-\sigma}(-\mathbf{p})} \omega_{\sigma}^{(+)}(\mathbf{p}) e^{-ipx} \frac{d\mathbf{p}}{2p^0}. \end{aligned}$$

Defining

$$\gamma^5 \stackrel{\text{def}}{=} i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i\alpha^1 \alpha^2 \alpha^3, \quad (4.106)$$

and noting that (4.93) and (4.68) imply

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & +\mathbb{1}_2 \end{pmatrix} \quad \text{in the representation (4.95),}$$

we get the following relations:

$$S(A^{-1}) \gamma^5 S(A) = \gamma^5, \quad (4.107)$$

$$[\gamma^{\mu}, \gamma^{\nu}]_{+} = \begin{cases} +2 & \text{if } \mu = \nu \in \{0, 5\}, \\ -2 & \text{if } \mu = \nu \in \{1, 2, 3\}, \\ 0 & \text{else,} \end{cases} \quad (4.108)$$

$$(\gamma^{\mu})^{*} = \begin{cases} +\gamma^{\mu} & \text{for } \mu \in \{0, 5\}, \\ -\gamma^{\mu} & \text{for } \mu \in \{1, 2, 3\}, \end{cases} \quad (4.109)$$

$$\hat{\mathbf{S}} \cdot \mathbf{e} \stackrel{\text{def}}{=} i \frac{d}{d\varphi} S(U_{\varphi \mathbf{e}})|_{\varphi=0} = \frac{1}{2} \gamma^5 \gamma^0 \boldsymbol{\gamma} \cdot \mathbf{e}, \quad (4.110)$$

$$[\hat{\mathbf{S}}, \gamma^5]_{-} = 0. \quad (4.111)$$

⁵¹However, because of anti-linearity of \mathfrak{T} , Schur's lemma is **not directly** applicable. Note that, for the same reason, $\mathfrak{T}^2 = \hat{\mathbb{1}}$ does not depend on the choice of phase factor.

⁵²In the representation (4.95): $i\gamma^1 \gamma^3 = -\begin{pmatrix} \tau^2 & 0 \\ 0 & \tau^2 \end{pmatrix}$.

Exercise 74 Show that

$$V_+^\mu(x) \stackrel{\text{def}}{=} \Psi^{(+)}(x)^* \gamma^0 \gamma^\mu \Psi^{(+)}(x) \quad (4.112)$$

is an ordinary current density and

$$V_-^\mu(x) \stackrel{\text{def}}{=} \Psi^{(+)}(x)^* \gamma^0 \gamma^5 \gamma^\mu \Psi^{(+)}(x) \quad (4.113)$$

an ordinary *axial* current density, i.e.⁵³

$$\begin{aligned} V_\pm^\mu(x^0, \mathbf{x}) &\xrightarrow{\hat{U}(a,A)} (\Lambda_A)^\mu{}_\nu V_\pm^\nu(\Lambda_A^{-1}(x-a)), \\ V_\pm^\mu(x^0, \mathbf{x}) &\xrightarrow{\hat{P}} \pm g_{\mu\nu} V_\pm^\nu(x^0, -\mathbf{x}), \\ V_\pm^\mu(x^0, \mathbf{x}) &\xrightarrow{\hat{T}} +g_{\mu\nu} V_\pm^\nu(-x^0, \mathbf{x}) \end{aligned} \quad (4.114)$$

Final remark: Physically relevant are only the **general** relations between the γ -matrices. Transformations of the type

$$\gamma^\mu \longrightarrow \gamma'^\mu = M \gamma^\mu M^{-1},$$

where M is any unitary 4×4 -matrix, are always allowed.⁵⁴ E.g., for

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} +\mathbb{1}_2 & +\mathbb{1}_2 \\ -\mathbb{1}_2 & +\mathbb{1}_2 \end{pmatrix}, \quad M^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} +\mathbb{1}_2 & -\mathbb{1}_2 \\ +\mathbb{1}_2 & +\mathbb{1}_2 \end{pmatrix}$$

we get the standard representation

$$\gamma'^0 = -\gamma^5, \quad \gamma'^5 = \gamma^0, \quad \gamma'^j = \gamma^j \quad \text{for } j = 1, 2, 3 \quad (4.115)$$

of (Bjorken and Drell, 1964).

The Limit $\frac{m}{|\mathbf{p}|} \rightarrow 0$

According to (4.110), (4.99) is equivalent to

$$\gamma^5 \hat{\mathbf{S}} \cdot \frac{\mathbf{p}}{|\mathbf{p}|} \hat{\Psi}^{(+)}(\mathbf{p}) = \frac{1}{2|\mathbf{p}|} (\omega_{\mathbf{p}} - m\gamma^0) \hat{\Psi}^{(+)}(\mathbf{p}). \quad (4.116)$$

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⁵³A more complete listing is given in (Itzykson and Zuber, 1980b, Sect. 3-4-4).

⁵⁴Of course, $S(A)$ (recall Exercise 71) has to be defined accordingly, as well as the $\omega_\pm^{(+)}(0)$:

$$\gamma^0 \omega_\pm^{(+)}(0) = \omega_\pm^{(+)}(0), \quad \hat{S}^3 \omega_\pm^{(+)}(0) = \pm \frac{1}{2} \omega_\pm^{(+)}(0).$$

By (4.108) and (4.111) the latter is equivalent to validity of the following two equations:⁵⁵

$$\begin{aligned}\hat{h}(1 + \gamma^5) \hat{\Psi}^{(+)}(\mathbf{p}) &= +\frac{\omega_{\mathbf{p}}}{2|\mathbf{p}|} (1 + \gamma^5) \hat{\Psi}^{(+)}(\mathbf{p}) - \frac{m}{2|\mathbf{p}|} \gamma^0 (1 - \gamma^5) \hat{\Psi}^{(+)}(\mathbf{p}), \\ \hat{h}(1 - \gamma^5) \hat{\Psi}^{(+)}(\mathbf{p}) &= -\frac{\omega_{\mathbf{p}}}{2|\mathbf{p}|} (1 - \gamma^5) \hat{\Psi}^{(+)}(\mathbf{p}) + \frac{m}{2|\mathbf{p}|} \gamma^0 (1 + \gamma^5) \hat{\Psi}^{(+)}(\mathbf{p}).\end{aligned}\quad (4.117)$$

This implies

$$\left. \begin{aligned}\hat{h} \hat{\Psi}_{\mathbf{R}}^{(+)}(\mathbf{p}) &= +\frac{1}{2} \hat{\Psi}_{\mathbf{R}}^{(+)}(\mathbf{p}) \\ \hat{h} \hat{\Psi}_{\mathbf{L}}^{(+)}(\mathbf{p}) &= -\frac{1}{2} \hat{\Psi}_{\mathbf{L}}^{(+)}(\mathbf{p})\end{aligned}\right\} \text{for } \frac{m}{|\mathbf{p}|} \rightarrow 0, \quad (4.118)$$

where

$$\begin{aligned}\hat{\Psi}_{\mathbf{R}}^{(+)}(\mathbf{p}) &\stackrel{\text{def}}{=} \frac{1}{2}(1 + \gamma^5) \hat{\Psi}^{(+)}(\mathbf{p}), \\ \hat{\Psi}_{\mathbf{L}}^{(+)}(\mathbf{p}) &\stackrel{\text{def}}{=} \frac{1}{2}(1 - \gamma^5) \hat{\Psi}^{(+)}(\mathbf{p}).\end{aligned}$$

Exercise 75 Show that

$$\gamma^0 \gamma^\mu = \left(\frac{1 + \gamma^5}{2} \right)^* \gamma^0 \gamma^\mu \frac{1 + \gamma^5}{2} + \left(\frac{1 - \gamma^5}{2} \right)^* \gamma^0 \gamma^\mu \frac{1 - \gamma^5}{2}$$

and hence

$$V_+^\mu(x) = \Psi_{\mathbf{R}}^{(+)}(x)^* \gamma^0 \gamma^\mu \Psi_{\mathbf{R}}^{(+)}(x) + \Psi_{\mathbf{L}}^{(+)}(x)^* \gamma^0 \gamma^\mu \Psi_{\mathbf{L}}^{(+)}(x).$$

Note that, in the representation (4.95), the equations

$$\hat{\Psi}_{\mathbf{R}}^{(+)}(\mathbf{p}) = \begin{pmatrix} 0 \\ \chi_{\mathbf{R}}(\mathbf{p}) \end{pmatrix}, \quad \hat{\Psi}_{\mathbf{L}}^{(+)}(\mathbf{p}) = \begin{pmatrix} \chi_{\mathbf{L}}(\mathbf{p}) \\ 0 \end{pmatrix},$$

with corresponding **2-component** spinors $\chi_{\mathbf{R}}(\mathbf{p})$, $\chi_{\mathbf{L}}(\mathbf{p})$, and

$$\hat{h} = \frac{1}{2} \begin{pmatrix} \boldsymbol{\tau} \cdot \frac{\mathbf{p}}{|\mathbf{p}|} & 0 \\ 0 & \boldsymbol{\tau} \cdot \frac{\mathbf{p}}{|\mathbf{p}|} \end{pmatrix}$$

hold. Hence, for $m = 0$, the equations (4.118) are equivalent to the two so-called **WEYL equations**

$$\begin{aligned}\partial_0 \Phi_{\mathbf{R}}(x) &= -\boldsymbol{\tau} \cdot \nabla_{\mathbf{x}} \Phi_{\mathbf{L}}(x), \quad \partial_0 \Phi_{\mathbf{L}}(x) = +\boldsymbol{\tau} \cdot \nabla_{\mathbf{x}} \Phi_{\mathbf{R}}(x), \\ \text{where: } \Phi_{\mathbf{R}(\mathbf{L})}(x) &\stackrel{\text{def}}{=} (2\pi)^{-\frac{3}{2}} \int_{p^0=\omega_{\mathbf{p}}} \chi_{\mathbf{R}(\mathbf{L})}(\mathbf{p}) e^{-ip^\mu x_\mu} \frac{d\mathbf{p}}{2p^0}.\end{aligned}\quad (4.119)$$

⁵⁵The first of these relations results by adding (4.116) and (4.116) multiplied by γ^5 (from the left). Recall that the helicity operator is $\hat{h} = \frac{\mathbf{p}}{|\mathbf{p}|} \cdot \hat{\mathbf{S}}$.

4.2.4 Quantized DIRAC Field

Positive Frequency Part of the DIRAC Field

Similarly to the electromagnetic field the DIRAC field is quantized by replacement of the amplitudes $b_\sigma(\mathbf{p})$ in (4.87) by corresponding annihilation operators $\hat{b}_\sigma(\mathbf{p})$ (of particles with linear momentum \mathbf{p} and \mathbf{e}_3 -component $\sigma \frac{1}{2}$ of the internal angular momentum in the **center of mass system** of the particle):

$$\hat{\Psi}^{(+)}(x) = \sqrt{2m} (2\pi)^{-\frac{3}{2}} \int_{p^0=\omega_{\mathbf{p}}} \underbrace{\sum_{\sigma=\pm} \hat{b}_\sigma(\mathbf{p}) \omega_\sigma^{(+)}(\mathbf{p})}_{\stackrel{\text{def}}{=} \hat{\Psi}^{(+)}(\mathbf{p})} e^{-ip^\mu x_\mu} \frac{d\mathbf{p}}{2p^0} \quad (4.120)$$

(compare (4.100)). Now, however, the $\hat{b}_\sigma(\mathbf{p})$ (respecting the PAULI *principle*) act in a space

$$\mathcal{H}_0 \stackrel{\text{def}}{=} \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}$$

the n -particle components $\mathcal{H}^{(n)}$ of which are spanned by totally **anti**-symmetric wave functions b_n :

$$b_n(\mathbf{p}_{\pi 1}, \sigma_{\pi 1}; \dots; \mathbf{p}_{\pi n}, \sigma_{\pi n}) = \text{sign}(\pi) b_n(\mathbf{p}_1, \sigma_1; \dots; \mathbf{p}_n, \sigma_n) \quad \text{for all } \pi \in S_n. \quad (4.121)$$

The inner product is given by natural generalization of the norm (4.88):

$$\|b_n\| \stackrel{\text{def}}{=} \sqrt{\sum_{\sigma_1, \dots, \sigma_n = \pm} \int \|b_n(\mathbf{p}_1, \sigma_1; \dots; \mathbf{p}_n, \sigma_n)\|^2 \prod_j \frac{d\mathbf{p}_j}{2\omega_{\mathbf{p}_j}}}. \quad (4.122)$$

Since, the $\hat{b}_\sigma(\mathbf{p})$ are given by

$$\begin{aligned} \left(\hat{b}_\sigma(\mathbf{p}) b\right)_0 &\stackrel{\text{def}}{=} b_1(\mathbf{p}, \sigma), \\ \left(\hat{b}_\sigma(\mathbf{p}) b\right)_{n-1}(\mathbf{p}_1, \sigma_1; \dots; \mathbf{p}_{n-1}, \sigma_{n-1}) &\stackrel{\text{def}}{=} \sqrt{n} b_n(\mathbf{p}, \sigma; \mathbf{p}_1, \sigma_1; \dots; \mathbf{p}_{n-1}, \sigma_{n-1}) \end{aligned} \quad (4.123)$$

on their natural domain of definition⁵⁶ we now have the **anti**-commutation relations

$$\begin{aligned} \left[\hat{b}_\sigma(\mathbf{p}), \hat{b}_{\sigma'}(\mathbf{p}')^*\right]_+ &= 2\omega_{\mathbf{p}} \delta_{\sigma\sigma'} \delta(\mathbf{p} - \mathbf{p}'), \\ \left[\hat{b}_\sigma(\mathbf{p}), \hat{b}_{\sigma'}(\mathbf{p}')\right]_+ &= 0. \end{aligned} \quad (4.124)$$

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⁵⁶The natural invariant domain of definition for $\hat{b}_\sigma(\mathbf{p})$ is characterized by the conditions

$$b_n^{\sigma_1, \dots, \sigma_n}(\mathbf{p}_1, \dots, \mathbf{p}_n) \stackrel{\text{def}}{=} b_n(\mathbf{p}_1, \sigma_1; \dots; \mathbf{p}_n, \sigma_n) \in \mathcal{S}(\mathbb{R}^{3n}) \quad \text{for fixed } (\sigma_1, \dots, \sigma_n) \in \{+, -\}^n$$

and

$$b_n = 0 \quad \text{for sufficiently large } n.$$

All linear relations of the 1-particle theory also hold for the quantized field. The field operator for the current density produced by all **particles** (not anti-particles) of charge⁵⁷ q is

$$\hat{j}_q^{(+)\mu}(x) \stackrel{\text{def}}{=} q \hat{\Psi}^{(+)}(x)^* \gamma^0 \gamma^\mu \hat{\Psi}^{(+)}(x) \quad (4.125)$$

and transforms like a 4-vector field (compare Exercise 74) under the natural extension $\hat{U}_0(a, A)$ of $\hat{U}(a, A)$ to all of \mathcal{H}_0 , fixed – together with the representation of space-time reflections – by the requirements

$$\hat{U}_0(a, A)^{-1} \hat{\Psi}^{(+)}(x) \hat{U}_0(a, A) = S(A) \hat{\Psi}^{(+)}(\Lambda_{A^{-1}}(x - a)), \quad (4.126)$$

$$\mathfrak{P}^{-1} \hat{\Psi}^{(+)}(x) \mathfrak{P} = \gamma^0 \hat{\Psi}^{(+)}(x^0, -\mathbf{x}), \quad (4.127)$$

$$\mathfrak{T}^{-1} \hat{\Psi}^{(+)}(x) \mathfrak{T} = i \gamma^1 \gamma^3 \hat{\Psi}^{(+)}(-x^0, \mathbf{x})^{*\text{T}}, \quad (4.128)$$

and

$$\hat{U}_0(a, A) \Omega_0 = \mathfrak{P} \Omega_0 = \mathfrak{T} \Omega_0 = \Omega_0 \left(\stackrel{\text{def}}{=} 1 \in \mathcal{H}^{(0)} = \mathbb{C} \right). \quad (4.129)$$

consistent with the described 1-particle theory. Due to⁵⁸

$$\omega_\sigma^{(+)}(\mathbf{p})^* \gamma^0 \omega_{\sigma'}^{(+)}(\mathbf{p}) = \delta_{\sigma\sigma'} \quad (4.130)$$

and

$$\int \hat{\Psi}^{(+)}(x)^* \gamma^0 \gamma^\mu \hat{\Psi}^{(+)}(x) d\mathbf{x} = \int_{p^0=\omega_{\mathbf{p}}} \hat{\Psi}^{(+)}(\mathbf{p})^* \gamma^0 \hat{\Psi}^{(+)}(\mathbf{p}) \frac{d\mathbf{p}}{2p^0}$$

(compare proof of (4.102)) the corresponding total charge is

$$\hat{Q}^{(+)} \stackrel{\text{def}}{=} \int \hat{j}_q^{(+)\mu}(x) d\mathbf{x} = q \int_{p^0=\omega_{\mathbf{p}}} \sum_{\sigma=\pm} \hat{b}_\sigma^*(\mathbf{p}) \hat{b}_\sigma(\mathbf{p}) \frac{d\mathbf{p}}{2p^0}, \quad (4.131)$$

independent of $x^0 \in \mathbb{R}$.

The theory with only positive frequencies has (among others) the following disadvantages:

(P1): The current density (4.125) violates Einstein's causality principle, i.e:

$$x \times y \not\Rightarrow \left[\hat{j}_q^{(+)\alpha}(x), \hat{j}_q^{(+)\beta}(y) \right]_- = 0.$$

(P2): In general, minimal coupling with an exterior field does not allow a solution for which both the incoming and the outgoing free DIRAC field have vanishing negative frequency parts.⁵⁹

⁵⁷ q need not be the electric charge!

⁵⁸(4.130) is trivial for $\mathbf{p} = 0$ and therefore, by (4.89) and (4.97), also valid for $\mathbf{p} \neq 0$.

⁵⁹This is related to KLEIN's *paradox* (see (Telegdi, 1995) and references given there).

Local Operator Field

Obviously problem (P2) requires the additional introduction of negative frequency solutions $\hat{\Psi}^{(-)}(x)$ of the DIRAC field operator equation. These can be constructed using the 4-spinors

$$\omega_{\sigma}^{(-)}(\mathbf{p}) \stackrel{\text{def}}{=} \gamma^5 \omega_{-\sigma}^{(+)}(-\mathbf{p}). \quad (4.132)$$

Exercise 76 Show the following:⁶⁰

1.

$$\omega_{+}^{(-)}(\mathbf{p}) = S\left(H_{\frac{\mathbf{p}}{\omega_{\mathbf{p}}}}\right)^{-1} \begin{pmatrix} 0 \\ -1 \\ 0 \\ +1 \end{pmatrix} = S\left(H_{\frac{\mathbf{p}}{\omega_{\mathbf{p}}}}\right)^{-1} \begin{pmatrix} -1 \\ 0 \\ +1 \\ 0 \end{pmatrix}, \quad \omega_{-}^{(-)}(\mathbf{p}) \quad (4.133)$$

in the representation (4.95).

2.

$$S\left(H_{\frac{\mathbf{p}}{\omega_{\mathbf{p}}}}\right) S\left(H_{\frac{\mathbf{p}}{\omega_{\mathbf{p}}}}\right) = \frac{p_{\mu} \gamma^{\mu}}{m} \gamma^0.$$

3.

$$\sum_{\sigma, \sigma' = \pm} \left(\omega_{\sigma}^{\sigma'}(\sigma' \mathbf{p})\right)_r \overline{\left(\omega_{\sigma}^{\sigma'}(\sigma' \mathbf{p})\right)_{r'}} = \left(\frac{p_{\mu} \gamma^{\mu}}{m} \gamma^0\right)_{r, r'}, \quad \text{with } p^0 = \omega_{\mathbf{p}}.$$

4.

$$\sum_{\sigma, \sigma' = \pm} \sigma' \left(\omega_{\sigma}^{\sigma'}(\sigma' \mathbf{p})\right)_r \overline{\left(\omega_{\sigma}^{\sigma'}(\sigma' \mathbf{p})\right)_{r'}} = (\gamma^0)_{r, r'}.$$

Then

$$\hat{\Psi}^{(-)}(x) = \sqrt{2m} (2\pi)^{-\frac{3}{2}} \int_{p^0 = -\omega_{\mathbf{p}}} \overbrace{\sum_{\sigma = \pm} \hat{d}_{\sigma}^{*}(-\mathbf{p}) \omega_{\sigma}^{(-)}(\mathbf{p})}^{\hat{\Psi}^{(-)}(\mathbf{p}) \stackrel{\text{def}}{=}} e^{-ip^{\mu} x_{\mu}} \frac{d\mathbf{p}}{2\omega_{\mathbf{p}}} \quad (4.134)$$

is indeed a solution of the DIRAC equation:

$$(\gamma^{\mu} p_{\mu} - m) \hat{\Psi}^{(-)}(\mathbf{p})|_{p^0 = -\omega_{\mathbf{p}}} = 0.$$

⁶⁰**Hint:** To prove the second statement, exploit (4.94) for the special case $\mathbf{p} = 0$, $A = H_{\frac{\mathbf{p}}{\omega_{\mathbf{p}}}}$.

Sketch of proof:

$$\begin{aligned}
& S\left(H_{\frac{\mathbf{p}}{\omega_{\mathbf{p}}}}\right) (\gamma^\mu p_\mu - m) |_{p^0 = -\omega_{\mathbf{p}}} \omega_\sigma^{(-)}(\mathbf{p}) \\
& \stackrel{(4.133)}{=} S\left(H_{\frac{\mathbf{p}}{\omega_{\mathbf{p}}}}\right) (\gamma^\mu p_\mu - m) |_{p^0 = -\omega_{\mathbf{p}}} S\left(H_{\frac{-\mathbf{p}}{\omega_{\mathbf{p}}}}\right) \omega_\sigma^{(-)}(0) \\
& \stackrel{(4.96)}{=} \left(\left(\Lambda_{H_{\frac{-\mathbf{p}}{\omega_{\mathbf{p}}}}} \right)^\mu{}_\nu \gamma^\nu p_\mu - m \right) |_{p^0 = -\omega_{\mathbf{p}}} \omega_\sigma^{(-)}(0) \\
& = - \left(\gamma^\mu \left(\Lambda_{H_{\frac{-\mathbf{p}}{\omega_{\mathbf{p}}}}(\omega_{\mathbf{p}}, -\mathbf{p}) \right)_\mu + m \right) \omega_\sigma^{(-)}(0) \\
& = -m (\gamma^0 + 1) \omega_\sigma^{(-)}(0) \\
& = 0. \quad \blacksquare
\end{aligned}$$

With (4.132) and (4.132) one can show that the equations (4.126)–(4.128) hold also for

$$\hat{\Psi}(x) \stackrel{\text{def}}{=} \hat{\Psi}^{(+)}(x) + \hat{\Psi}^{(-)}(x) \quad (4.135)$$

instead of $\hat{\Psi}^{(+)}(x)$ if the

$$\check{b}_\sigma(\mathbf{p}) \stackrel{\text{def}}{=} \sigma \hat{d}_\sigma(\mathbf{p})$$

transform the same way as the $\hat{b}_\sigma(\mathbf{p})$ do.

Hence, in principle, one could identify the operators $\check{b}_\sigma(\mathbf{p})$ with the operators $\hat{b}_\sigma(\mathbf{p})$. However, a solution of the problems mentioned above will be achieved⁶¹ only if the $\hat{d}_\sigma(\mathbf{p})$ are interpreted as annihilation operators for **anti**-particles, with invariant domain $\check{D}_0 \subset \check{\mathcal{H}}_0$ (compare Section 2.4.1). Then the operators

$$\hat{b}_\sigma(\mathbf{p}) = \hat{b}_\sigma(\mathbf{p}) \otimes \hat{1}, \quad \hat{d}_\sigma(\mathbf{p}) = (-1)^{\hat{Q}^{(+)/q}} \hat{1} \otimes \hat{d}_\sigma(\mathbf{p}),$$

well-defined on

$$D_D \stackrel{\text{def}}{=} D_0 \otimes \check{D}_0 \subset \mathcal{H}_D \stackrel{\text{def}}{=} \mathcal{H}_0 \otimes \check{\mathcal{H}}_0,$$

fulfill the anti-commutation relations

$$\left[\hat{d}_\sigma(\mathbf{p}), \hat{b}_{\sigma'}(\mathbf{p}') \right]_+ = \left[\hat{d}_\sigma(\mathbf{p})^*, \hat{b}_{\sigma'}(\mathbf{p}') \right]_+ = 0 \quad (4.136)$$

in addition to (4.124) and the corresponding relations for the $\hat{d}_\sigma(\mathbf{p})$. From this, using the results of Exercise 76, one easily derives the anti-commutation relations

$$\begin{aligned}
& \left[\left(\hat{\Psi}^{(\sigma)}(x) \right)_r, \left(\hat{\Psi}^{(\sigma')}(x') \right)_{r'} \right]_+ = 0, \\
& \left[\left(\hat{\Psi}^{(\sigma)}(x) \right)_r, \left(\hat{\Psi}^{(\sigma')}(x') \right)_{r'}^* \right]_+ = \delta_{\sigma\sigma'} \left((i\gamma^\mu \partial_\mu + m) \gamma^0 \right)_{rr'} i\Delta_m^{(\sigma)}(x - x').
\end{aligned} \quad (4.137)$$

Thus the solution (4.135) of the DIRAC equation

$$(i\gamma^\mu \partial_\mu - m) \hat{\Psi}(x) = 0, \quad (4.138)$$

⁶¹See, e.g., (Seiler, 1978).

is a *local* FERMION *field*, i.e.:

$$x \times x' \implies \left[\left(\hat{\Psi}(x) \right)_r, \left(\hat{\Psi}(x') \right)_{r'}^{(*)} \right]_+ = 0. \quad (4.139)$$

Moreover, the anti-commutation relations (4.137) imply that

$$\hat{j}_q^\mu(x) \stackrel{\text{def}}{=} q : \hat{\Psi}(x)^* \gamma^0 \gamma^\mu \hat{\Psi}(x) : \quad (4.140)$$

is a *local* BOSE *field*, i.e.:

$$x \times x' \implies \left[\hat{j}_q^\alpha(x), \hat{j}_q^\beta(x') \right]_- = 0. \quad (4.141)$$

Normal ordering $::$ in (4.140) means that FERMION creation and annihilation operators have to be **anti**-commute, if necessary, irrespective of the actual anti-commutation relations until no creation operator is on the right of any annihilation operator.⁶²

$\hat{j}_q^\mu(x)$ is interpreted as observable of the total current density since, e.g., the following relations hold:

$$\int \hat{j}_q^0(x) \, d\mathbf{x} = \hat{Q}_q \stackrel{\text{def}}{=} q \int_{p^0=\omega_{\mathbf{p}}} \sum_{\sigma=\pm} \left(\hat{b}_\sigma^*(\mathbf{p}) \hat{b}_\sigma(\mathbf{p}) - \hat{d}_\sigma^*(\mathbf{p}) \hat{d}_\sigma(\mathbf{p}) \right) \frac{d\mathbf{p}}{2p^0}, \quad (4.142)$$

$$\partial_\mu \hat{j}_q^\mu(x) = 0, \quad (4.143)$$

$$\hat{j}_q^\mu(x) = \hat{j}_q^\mu(x)^*. \quad (4.144)$$

Since

$$\Delta_m(0, \mathbf{x}) = 0, \quad (\partial_0 \Delta(x))|_{x^0=0} = -\delta(\mathbf{x}), \quad (4.145)$$

the anti-commutation relations (4.137) imply the canonical anti-commutation relations

$$\begin{aligned} \left[\left(\hat{\Psi}(x^0, \mathbf{x}) \right)_r, \left(\hat{\Psi}(x^0, \mathbf{x}') \right)_{r'} \right]_+ &= 0, \\ \left[\left(\hat{\Psi}(x^0, \mathbf{x}) \right)_r, \left(\hat{\Psi}(x^0, \mathbf{x}') \right)_{r'}^* \right]_+ &= \delta_{rr'} \delta(\mathbf{x} - \mathbf{x}'). \end{aligned} \quad (4.146)$$

Therefore,⁶³

$$\hat{P}_D^\mu \stackrel{\text{def}}{=} \int : \hat{\Psi}(x)^* i \partial^\mu \hat{\Psi}(x) : \, d\mathbf{x} \quad (4.147)$$

is the generator of space-time translations, i.e.

$$i \left[\hat{P}_D^\mu, \hat{\Psi}(x) \right]_- = \partial^\mu \hat{\Psi}(x). \quad (4.148)$$

⁶²Another effect of normal ordering, besides making $\hat{j}_q^\mu(x)$ well-defined (as operator-valued distribution, not just a quadratic form), is that the quantized current density – contrary to the classical one – is no longer positive (compare remark on Corollary 2.2.14).

⁶³Note that $[\hat{A}\hat{B}, \hat{C}]_- = \hat{A}[\hat{B}, \hat{C}]_+ - [\hat{A}, \hat{C}]_+ \hat{B}$.

Final remark: Note that the **local** DIRAC formalism is physically consistent only since anti-particles actually exist and are different from the corresponding particles (e.g. having charge of different sign). Contrary to this, local BOSE fields may well describe particles (e.g. photons) that are identical with their anti-particles.

The Limit $\frac{m}{|\mathbf{p}|} \rightarrow 0$

Particle or anti-particles in states which can be created by applying (smeared versions of)

$$\hat{\Psi}_R(x) = \frac{1}{2}(1 + \gamma^5)\hat{\Psi}(x) \quad (4.149)$$

or $\hat{\Psi}_R(x)^*$ to the vacuum vector are called **right handed**. Similarly, particle or anti-particles in states which can be created by applying (smeared versions of)

$$\hat{\Psi}_L(x) = \frac{1}{2}(1 - \gamma^5)\hat{\Psi}(x) \quad (4.150)$$

or $\hat{\Psi}_L(x)^*$ to the vacuum vector are called **left handed**.

The transition $\frac{m}{|\mathbf{p}|} \rightarrow 0$ may be performed as discussed at the end of 4.2.3:

$$\begin{aligned} \text{for } m = 0 : \quad & \left[\hat{\Psi}_R(x), \hat{h} \right]_+ = +\frac{1}{2}\hat{\Psi}_R(x) \\ & \left[\hat{\Psi}_L(x), \hat{h} \right]_+ = -\frac{1}{2}\hat{\Psi}_L(x) \end{aligned} \quad (4.151)$$

where now: $\hat{h} \stackrel{\text{def}}{=} \int dx : \hat{\Psi}(x)^* \left(\frac{1}{2}\gamma^5\gamma^0\gamma^j \right) \frac{i\partial^j}{|\nabla_x|} \hat{\Psi}(x) : .$

(compare (4.110) and (4.117)). This implies:

$$\begin{aligned} \text{helicity of } \textit{right handed} \text{ massless particles:} & \quad +1/2 \\ \text{helicity of } \textit{left handed} \text{ massless particles:} & \quad -1/2 \\ \text{helicity of } \textit{left handed} \text{ massless } \mathbf{anti-} \text{ particles:} & \quad +1/2 \\ \text{helicity of } \textit{right handed} \text{ massless } \mathbf{anti-} \text{ particles:} & \quad -1/2 \end{aligned} \quad (4.152)$$

4.3 The S-Matrix of Quantum Electrodynamics (QED)

Let us choose HEAVISIDE units (in addition to $\hbar = c = 1$). Then, by (4.51) and Footnote 25,

$$\zeta = 1. \quad (4.153)$$

4.3.1 Naive Interaction Picture of QED

Asymptotic Description

For simplicity we consider the interaction of the electromagnetic field with the electron-positron field, only. Then, in the GUPTA-BLEULER formalism, the interacting system of quantum electrodynamics is asymptotically identified (in the sense of 2.3.1) with the following ‘free’ system:

The basic HILBERT space (containing unphysical degrees of freedom) is

$$\mathcal{H} = \mathcal{H}_D \otimes \mathcal{H}_{GB}, \quad (4.154)$$

where \mathcal{H}_D is the HILBERT space of the free electron-positron system as described in 4.2.4 and \mathcal{H}_{GB} is the HILBERT space of the GUPTA-BLEULER description of the quantized free electromagnetic field given in 4.1.3. The DIRAC field $\hat{\Psi}(x)$ will now be identified with $\hat{\Psi}(x) \otimes \hat{1}$ and the GUPTA-BLEULER field $\hat{A}_{GB}^\mu(x)$ with $\hat{1} \otimes \hat{A}_{GB}^\mu(x)$. Then the vacuum state vector

$$\Omega = \Omega_D \otimes \Omega_{GB}$$

of the total system is cyclic w.r.t. the fields $\hat{A}_{GB}^\mu(x)$, $\hat{\Psi}(x)_r$ and $\hat{\Psi}(x)_l^\dagger$, well-defined as tempered field operators with invariant domain⁶⁴

$$D = D_D \otimes D_A.$$

Here, of course, $\Omega_D \stackrel{\text{def}}{=} \Omega_0 \otimes \check{\Omega}_0$ denotes the vacuum state vector of the DIRAC theory and Ω_{GB} the vacuum state vector of the GUPTA-BLEULER formalism (also denoted by Ω in 4.1.3). The new vacuum state vector is invariant under the representation

$$\hat{U}(a, A) = \hat{U}_D(a, A) \otimes \hat{V}(a, \Lambda_A)$$

of $i\text{SL}(2, \mathbb{C})$ (compare 4.2.2), where $\hat{U}_D(a, A) \stackrel{\text{def}}{=} \hat{U}_0(a, A) \times \check{\check{U}}_0(a, A)$ denotes the representation of $i\text{SL}(2, \mathbb{C})$ for the DIRAC theory and $\hat{V}(a, \Lambda)$ the representation of \mathcal{P}_+^\dagger given in 4.1.3. It is unitary w.r.t. the indefinite inner product

$$(\Phi_1 | \Phi_2) \stackrel{\text{def}}{=} \langle \Phi_1 | \hat{1} \otimes \hat{\eta} \Phi_2 \rangle. \quad (4.155)$$

with $\hat{\eta}$ defined by (4.29).

The subset of D describing physical states is

$$D_F = \left\{ \Phi \in D : \overline{\partial_\mu \hat{A}_{GB}^{\mu(+)}(x) \Phi} = 0 \right\}$$

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⁶⁴Here \otimes denotes the algebraic tensor product whereas in (4.154), of course, the topological tensor product of HILBERT spaces has to be taken.

(compare (4.33)). Elements of D_F describe the same physical state if their difference is in

$$D_{00} = \{\Phi \in D_F : (\Phi | \Phi) = 0\}$$

(compare (4.17)). For **physical** states the expectation values⁶⁵ of the quantized **free** electromagnetic field

$$\hat{F}_{\text{GB}}^{\mu\nu}(x) = \partial^\mu \hat{A}_{\text{GB}}^\nu(x) - \partial^\nu \hat{A}_{\text{GB}}^\mu(x)$$

(compare (4.13)) fulfill the **free** MAXWELL equations

$$\partial_\mu \left(\Phi | \hat{F}_{\text{GB}}^{\mu\nu}(x) \Phi \right) = 0 \quad \forall \Phi \in D_F.$$

(compare final remark of 4.1.3) although the current density (4.27) of the free DIRAC field does not vanish on D_F . This means that the **free electromagnetic field operators describe only the radiative part** – not the field dragged along by the asymptotic charged particles (and contributing to their physical mass).

The Hamiltonian of the FS is

$$\hat{H}_0 = \hat{P}_D^0 + \hat{P}_{\text{GB}}^0 \quad (4.156)$$

with \hat{P}_D^0 resp. \hat{P}_{GB}^0 given by (4.147) resp. (4.48).

Formal Minimal Coupling

By (4.153) and (4.145), the commutation relations (4.27) imply the canonical commutation relations

$$\begin{aligned} \left[\hat{A}_{\text{GB}}^\mu(x), \left(\frac{\partial}{\partial x'^0} \hat{A}_{\text{GB}}^{\mu'}(x') \right) \Big|_{x'^0=x^0} \right]_- &= -i g^{\mu\mu'} \delta(\mathbf{x} - \mathbf{x}') \\ \left[\hat{A}_{\text{GB}}^\mu(x), \hat{A}_{\text{GB}}^\mu(x^0, \mathbf{x}') \right]_- &= \left[\frac{\partial}{\partial x'^0} \hat{A}_{\text{GB}}^{\mu'}(x), \left(\frac{\partial}{\partial x'^0} \hat{A}_{\text{GB}}^{\mu'}(x') \right) \Big|_{x'^0=x^0} \right]_- = 0. \end{aligned} \quad (4.157)$$

These and the canonical anti-commutation relations (4.146) for the DIRAC field (together with (4.27) and (4.137)) imply⁶⁶ that a **formal** solution of the fundamental differential equations

$$\begin{aligned} \square \hat{A}_{\text{int}}^\mu(x) &= \hat{j}_{\text{int}}^\mu(x) \stackrel{\text{def}}{=} e^{i\hat{H}x^0} \hat{j}_{-e}^\mu(0, \mathbf{x}) e^{-i\hat{H}x^0}, \\ \left(i\gamma^\mu \left(\partial_\mu - ie g_{\mu\nu} \hat{A}_{\text{int}}^\nu(x) \right) - m \right) \hat{\Psi}_{\text{int}}(x) &= 0, \end{aligned} \quad (4.158)$$

of QED is given by

$$\begin{aligned} \hat{A}_{\text{int}}^\mu(x) &\stackrel{\text{def}}{=} e^{i\hat{H}x^0} \hat{A}_{\text{GB}}^\mu(0, \mathbf{x}) e^{-i\hat{H}x^0}, \\ \hat{\Psi}_{\text{int}}(x) &\stackrel{\text{def}}{=} e^{i\hat{H}x^0} \hat{\Psi}(0, \mathbf{x}) e^{-i\hat{H}x^0}, \end{aligned} \quad (4.159)$$

⁶⁵Recall that expectation values are given via $(. | .)$, in the GUPTA-BLEULER formalism

⁶⁶Recall Section 3.1.2

with

$$\hat{H} = \hat{P}_D^0 + \hat{P}_{GB}^0 + \hat{H}_{\text{int}}, \quad (4.160)$$

where

$$\hat{H}_{\text{int}} = \int g_{\mu\nu} : \hat{j}_{-e}^\mu(0, \mathbf{x}) \hat{A}_{GB}^\nu(0, \mathbf{x}) : d\mathbf{x}. \quad (4.161)$$

Remark: Here, e is the modulus of the electron charge in HEAVISIDE **units** (see Appendix A.3.4 von (Lücke, edyn)). Note that the definition of $\hat{j}_{\text{int}}(x)$ is only a formal one. $\hat{j}_q^\mu(x)$ was defined in (4.140), but now $*$ has to be replaced by \dagger , of course. The transition from (4.138)/(4.9) to (4.158) is called *minimal coupling*.

Exercise 77 Show that

$$\left[\hat{j}_{-e}^\mu(0, \mathbf{x}'), \hat{\Psi}(0, \mathbf{x}) \right]_- = +e \delta(\mathbf{x} - \mathbf{x}') \gamma^0 \gamma^\mu \hat{\Psi}(0, \mathbf{x}).$$

By (4.156), the operator (3.9) is

$$\hat{H}_I(x^0) = \int g_{\mu\nu} : \hat{j}_{-e}^\mu(x) \hat{A}_{GB}^\nu(x) : d\mathbf{x},$$

i.e. we have to set⁶⁷

$$\boxed{\hat{S}_1(x) = ie g_{\mu\nu} : \hat{\Psi}(x) \dagger \gamma^0 \gamma^\mu \hat{\Psi}(x) \hat{A}_{GB}^\nu(x) :} \quad (4.162)$$

in (3.26) (and let $g \rightarrow 1$) **if** the time ordering is suitably defined.

Transition Probabilities

If the actual state of the IS looks for $t \rightarrow -\infty$ like the state of the FS described by Φ then the probability for a positive outcome of an ideal test whether the IS is in a state looking for $t \rightarrow +\infty$ like the state of the FS described by Φ' is the **transition probability**

$$p(\Phi \rightarrow \Phi') \stackrel{\text{def}}{=} \left| \left(\Phi' \mid \hat{S}_0 \Phi \right) \right|^2 \quad (4.163)$$

(recall Section 2.3.1). In practice one is only interested in states Φ, Φ' of the form

$$\int \hat{c}_{k_1}(\mathbf{p}_1)^\dagger \cdots \hat{c}_{k_N}(\mathbf{p}_N)^\dagger \varphi(\mathbf{p}_1, \dots, \mathbf{p}_N) d\mathbf{p}_1 \cdots d\mathbf{p}_N \Omega, \quad \varphi \in \mathcal{S}(\mathbb{R}^{3N}),$$

where the $\hat{c}_k(\mathbf{p})^\dagger$ are suitable creation operators. Then the essential task is to calculate the **scattering amplitudes**

$$\left(\hat{c}_{j_1}(\mathbf{p}'_1)^\dagger \cdots \hat{c}_{j_{N'}}(\mathbf{p}'_{N'})^\dagger \Omega \mid \left(\hat{S}_0 - \hat{1} \right) \hat{c}_{k_1}(\mathbf{p}_1)^\dagger \cdots \hat{c}_{k_N}(\mathbf{p}_N)^\dagger \Omega \right). \quad (4.164)$$

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⁶⁷Note that, by Lemma 4.1.3,

$$\Phi \in D_{00} \implies \int \hat{S}_1(x) g(x) dx \Phi \in \overline{D_{00}}^{(\cdot, \cdot)}$$

holds for all $g \in \mathcal{S}(\mathbb{R}^4)$.

4.3.2 General Perturbation Theory

Generalization of WICK's Theorem

To every type of field appearing in $\hat{S}_1(x)$ and its adjoint (if not identical to the field itself) one assign a characteristic type of line; e.g. wavy lines for photons, simple lines with upward orientation for electrons etc.

The *transition matrix*

$$\hat{S}_0 - \hat{1} = \sum_{n=1}^{\infty} \int T\left(\hat{S}_1(x_1) \cdots \hat{S}_1(x_n)\right) dx_1 \cdots dx_n \quad (4.165)$$

(compare (3.26)) is evaluated by first writing the expressions

$$T\left(\hat{S}_1(x_1) \cdots \hat{S}_1(x_n)\right) - \langle \Omega | T\left(\hat{S}_1(x_1) \cdots \hat{S}_1(x_n)\right) \Omega \rangle$$

as linear combinations of normally ordered products. This involves so-called *internal contractions* of pairs of field operators appearing in $\hat{S}_1(x_1) \cdots \hat{S}_1(x_n)$ and depending on different variables x_ν . These contractions will be characterized by joining typical lines attached the operators to be contracted. For instance, the (dashed) line in

$$:\hat{\Phi}_{11}(x_1) \cdots \hat{\Phi}_{j1}(x_1) \cdots : \hat{\Phi}_{12}(x_2) \cdots : \hat{\Phi}_{1n}(x_n) \cdots \hat{\Phi}_{kn}(x_n) \cdots :$$

|-----|

means internal contraction of the pair of field operators $\hat{\Phi}_{j1}(x_1), \hat{\Phi}_{kn}(x_n)$ (corresponding to the line type), i.e. this pair has to be replaced by its *propagator*⁶⁸

$$\left(\Omega \left| T\left(\hat{\Phi}_{j1}(x_1)\hat{\Phi}_{kn}(x_n)\right) \Omega \right. \right).$$

If $\hat{\Phi}_{j1}(x_1)$ and $\hat{\Phi}_{kn}(x_n)$ are both FERMION fields with an odd number of FERMION fields in between, a factor -1 has to be applied in addition. Here the time ordering T is defined as the least singular *covariant* operation fulfilling the condition

$$\boxed{\begin{aligned} T\left(\hat{\Phi}(x)\hat{\Phi}'(x')\right) &\stackrel{\text{def}}{=} \begin{cases} \hat{\Phi}(x)\hat{\Phi}'(x') & \text{for } x^0 > x'^0, \\ \sigma\hat{\Phi}'(x')\hat{\Phi}(x) & \text{for } x^0 < x'^0, \end{cases} \\ \text{where: } \sigma &\stackrel{\text{def}}{=} \begin{cases} -1 & \text{if both } \hat{\Phi} \text{ and } \hat{\Phi}' \text{ are FERMION fields,} \\ +1 & \text{else.} \end{cases} \end{aligned}} \quad (4.166)$$

⁶⁸The definition of the propagators implies that only contractions of pairs with fitting line types can be different from zero.

For simplicity we assume that $\hat{S}_1(x)$, which should be a BOSE field,⁶⁹ is a monomial⁷⁰ of field operators. Then one may prove⁷¹

$$\begin{aligned} & T\left(\hat{S}_1(x_1)\cdots\hat{S}_1(x_n)\right) - \langle \Omega | T\left(\hat{S}_1(x_1)\cdots\hat{S}_1(x_n)\right) \Omega \rangle \\ &= : \hat{S}_1(x_1)\cdots\hat{S}_1(x_n) : \\ &\quad + : \hat{S}_1(x_1)\cdots\hat{S}_1(x_j)\cdots\cdots\hat{S}_1(x_k)\cdots\hat{S}_1(x_n) : \\ &\quad\quad + \text{all other 1-fold contractions} \\ &\quad + : \cdots \hat{S}_1(x_{j_1})\cdots\hat{S}_1(x_{j_2})\cdots\cdots\hat{S}_1(x_{j_3})\cdots\hat{S}_1(x_{j_4})\cdots : \\ &\quad\quad\quad \underbrace{\hspace{10em}} \\ &\quad\quad\quad + \text{all other 2-fold contractions} \\ &\quad + : \cdots \hat{S}_1(x_{j_1})\cdots\hat{S}_1(x_{j_2})\cdots\cdots\hat{S}_1(x_{j_3})\cdots : \\ &\quad\quad\quad \underbrace{\hspace{10em}} \\ &\quad + \text{all other **incomplete** multiple contractions.} \end{aligned}$$

for pairwise different x_ν^0 by straightforward generalization of the techniques used in Section 3.2.1.

Let $\hat{K}(x_1, \dots, x_n)$ be any of the incompletely contracted $: \hat{S}_1(x_1)\cdots\hat{S}_1(x_n) :$ -terms (maybe $: \hat{S}_1(x_1)\cdots\hat{S}_1(x_n) :$ itself) and assume that products of propagators are suitably defined, if necessary.⁷² Then its contribution to (4.164) is evaluated by rewriting

$$\hat{c}_{j_{N'}}(\mathbf{p}'_{N'}) \cdots \hat{c}_{j_1}(\mathbf{p}'_1) \hat{K}(x_1, \dots, x_n) \hat{c}_{k_1}^\dagger(\mathbf{p}_1) \cdots \hat{c}_{k_N}^\dagger(\mathbf{p}_N) \tag{4.167}$$

as the $K(x_1, \dots, x_n; \mathbf{p}_1, \dots, \mathbf{p}_N; \mathbf{p}'_1, \dots, \mathbf{p}'_{N'})$ -fold of $\hat{1}$ plus some linear combination of normally ordered field products. Then, by (4.165), the contribution of $\hat{K}(x_1, \dots, x_n)$ to (4.164) is given by

$$\frac{1}{n!} \int K(x_1, \dots, x_n; \mathbf{p}_1, \dots, \mathbf{p}_N; \mathbf{p}'_1, \dots, \mathbf{p}'_{N'}) dx_1 \cdots dx_n,$$

where, by straightforward generalization of Wick's theorem,

$$\begin{aligned} & K(x_1, \dots, x_n; \mathbf{p}_1, \dots, \mathbf{p}_N; \mathbf{p}'_1, \dots, \mathbf{p}'_{N'}) \\ &= \underbrace{\hat{c}_{j_{N'}}(\mathbf{p}'_{N'}) \cdots \hat{c}_{j_1}(\mathbf{p}'_1) : \hat{K}(x_1, \dots, x_n) : \hat{c}_{k_1}^\dagger(\mathbf{p}_1) \cdots \hat{c}_{k_N}^\dagger(\mathbf{p}_N)}_{\text{---|---|---|---|}} \tag{4.168} \\ &\quad + \text{all other **complete** external contractions.} \end{aligned}$$

External contractions are those involving at least one of the operators $\hat{c}_k(\mathbf{p})^\dagger$ or $\hat{c}_j(\mathbf{p}')$ and for which the corresponding 2-point function (without time-ordering) is used instead of the propagator.

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⁶⁹Note that the time ordering of the $\hat{S}_1(x_\nu)$ -factors used in (3.26) resp. (4.165) was that for BOSE fields.

⁷⁰The necessary modifications for polynomials are obvious.

⁷¹As in 3.2.1, all field operators have to be formally considered as different in the contraction schemes. Recall $\langle \Omega | \hat{S}_0 \Omega \rangle = \langle \Omega | \Omega \rangle$.

⁷²Actually, as discussed in Section 3.2.2 for the $\lambda\hat{\Phi}^4$ -theory, physically correct definition of these products is the task of renormalization theory.

FEYNMAN Diagrams with External Momenta

The nontrivial terms contribution to (4.168) can be represented by FEYNMAN diagrams G of the following type:

1. G consists vertices (corresponding to the variables x_ν) numbered $1, \dots, V_G > 0$ and lines representing internal or external contractions, the free ends of the latter being provided with a unique characterization of the operator ($\hat{c}_k(\mathbf{p})^\dagger$ or $\hat{c}_j(\mathbf{p}')$) to be contracted.
2. Every vertex is connected with a family of contraction lines corresponding to the family of field factors building $\hat{S}_1(x)$.
3. Every line representing an internal contraction forms a direct link between two vertices.
4. Every line representing an external contraction has at least one free end.
5. A free end of a line representing an external contraction has to be the lowest point of this line if it corresponds to a creations operator.
6. A free end of a line representing an external contraction has to be the highest point of this line if it corresponds to an annihilation operator.

As demonstrated in 3.2.3 for the $\lambda\hat{\Phi}^4$ -theory one can show that the terms in (4.168) corresponding to diagrams with vacuum subdiagrams may be skipped.

The admitted diagrams are heuristically interpreted as follows:

- Every lower end of an exterior line represents an incoming particle corresponding to the attached information.
- Every higher end of an exterior line represents an outgoing particle corresponding to the attached information.
- Every internal line represent a *virtual* particle corresponding to the line type.
- Every vertex represents an event where the particles corresponding to the connected lines interact with each other being annihilated or created respectively.

The four diagrams of QED sketched in Figure 4.1, e.g., describe scattering of two electrons with incoming momenta $\mathbf{p}_1, \mathbf{p}_2$ and outgoing momenta $\mathbf{p}'_1, \mathbf{p}'_2$.

Similarly, the four diagrams of Figure 4.2 describe scattering of an electron having initial momentum \mathbf{p}_1 and final momentum \mathbf{p}'_1 with a positron having initial momentum \mathbf{p}_2 and final momentum \mathbf{p}'_2 .

⁷³Actually, the spin states should also be indicated at the ends of the external lines.

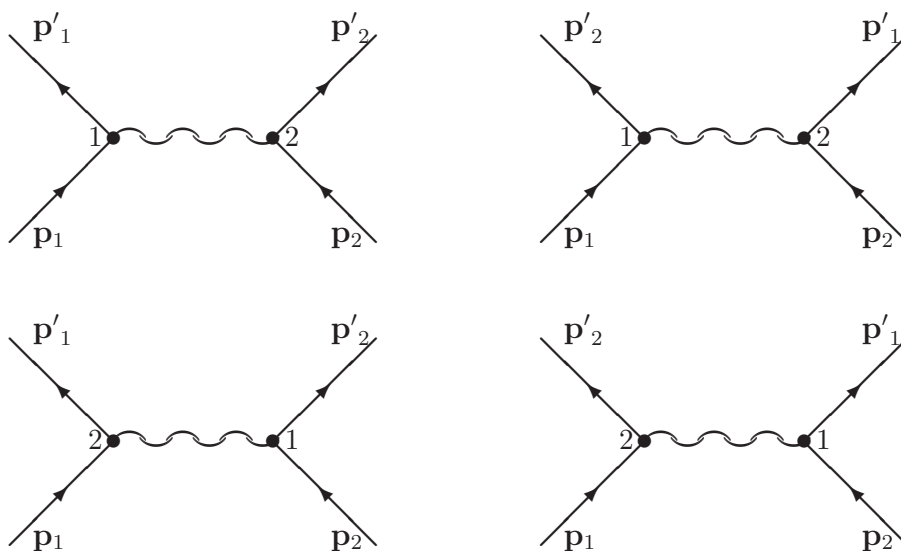


Figure 4.1: MØLLER scattering⁷³

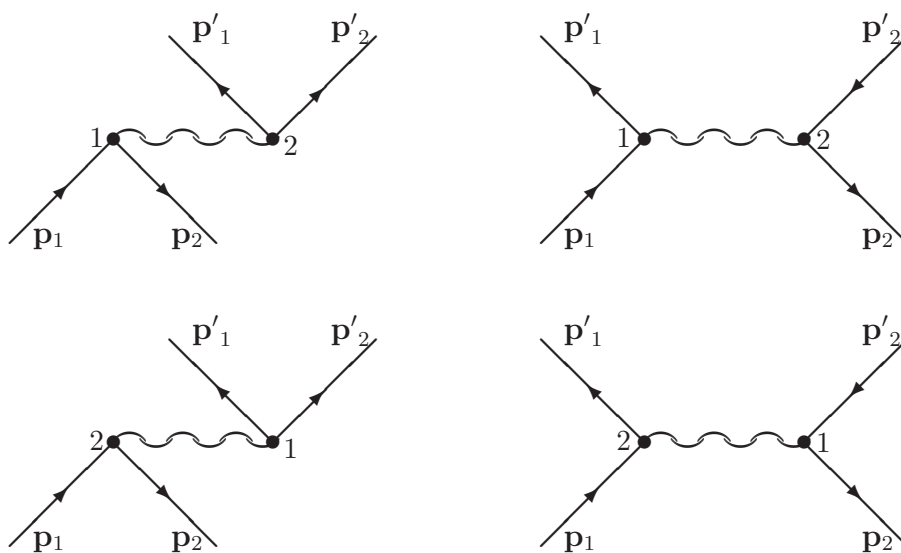
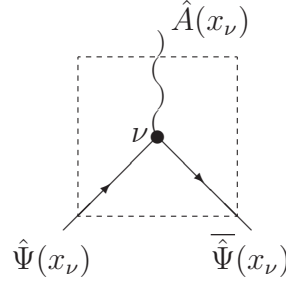


Figure 4.2: BHABBA scattering

Figure 4.3: A typical⁷⁵ vertex of QED

$$\begin{aligned}
 j \bullet \longleftarrow \bullet k &\cong \left\langle \Omega \mid T \left(\left(\hat{\Psi}(x_j) \right)_{r_j} \left(\hat{\Psi}(x_k)^\dagger \gamma^0 \right)_{l_k} \right) \Omega \right\rangle \\
 &\stackrel{\text{def}}{=} +i(2\pi)^{-4} \lim_{\epsilon \rightarrow +0} \int dq \frac{(\gamma^\mu q_\mu + m)_{r_j l_k}}{q^2 - m^2 + i\epsilon} e^{-iq(x_j - x_k)} \\
 j \bullet \longrightarrow \bullet k &\cong \left\langle \Omega \mid T \left(\left(\hat{\Psi}(x_j)^\dagger \gamma^0 \right)_{l_j} \left(\hat{\Psi}(x_k) \right)_{r_k} \right) \Omega \right\rangle \\
 &\stackrel{\text{def}}{=} -i(2\pi)^{-4} \lim_{\epsilon \rightarrow +0} \int dq \frac{(\gamma^\mu q_\mu + m)_{r_k l_j}}{q^2 - m^2 + i\epsilon} e^{-iq(x_k - x_j)} \\
 j \bullet \text{---} \bullet k &\cong \left\langle \Omega \mid T \left(\hat{A}^{\mu_j}(x_j) \hat{A}^{\mu_k}(x_k) \right) \Omega \right\rangle \\
 &\stackrel{\text{def}}{=} -g^{\mu_j \mu_k} i(2\pi)^{-4} \lim_{\epsilon \rightarrow +0} \int \frac{dk}{k^\nu k_\nu + i\epsilon} e^{\pm ik(x_j - x_k)}
 \end{aligned}$$

Figure 4.4: Internal Lines⁷⁶ of QED ($j < k$)

4.3.3 The FEYNMAN Rules of QED

In QED, i.e. if $\hat{S}_1(x)$ is given by (4.162), every vertex is connected with exactly three lines:⁷⁴ a wavy photon line corresponding to \hat{A}_{GB} , an incoming solid fermion line corresponding to $\hat{\Psi}$, and an outgoing solid fermion line corresponding to $\bar{\hat{\Psi}} = \hat{\Psi}^\dagger \gamma^0$. This is sketched in Figure 4.3.

The propagators corresponding to internal lines connecting the indices j and $k > j$ are those given by Figure 4.4. If one is interested only in **linearly polarized** asymptotic ‘particles’ then the $\hat{c}_j(\mathbf{p})$ used in (4.164) are DIRAC operators

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⁷⁴Actually, since $\hat{S}_1(x)$ is a sum of monomials (recall Footnote 70) should be considered as a family of lines with indices to be summed over according to the FEYNMAN rules formulated below.

⁷⁵The orientation is relevant only for external lines.

⁷⁶Die notation r_j, l_k for the indices is to indicate the original position (left/right) of the contracted field operators relative to each other. For the definition of the bosonic propagator recall (3.28)/(3.30) and (4.26). For the fermionic propagator recall (4.137).

$\hat{b}_\sigma(\mathbf{p}), \hat{d}_\sigma(\mathbf{p})$ (compare (4.2.4)) or **transversal** photon operators⁷⁷

$$\hat{a}_\epsilon(\mathbf{p}) \stackrel{\text{def}}{=} \boldsymbol{\epsilon}(\mathbf{p}) \cdot \sum_{j=1}^3 \mathbf{e}_j \hat{a}^j(\mathbf{p})$$

with

$$\epsilon^0(\mathbf{p}) = 0, \quad \boldsymbol{\epsilon}(\mathbf{p}) = \boldsymbol{\epsilon}(\mathbf{p})^*, \quad |\boldsymbol{\epsilon}(\mathbf{p})| = 1, \quad \mathbf{p} \cdot \boldsymbol{\epsilon}(\mathbf{p}) = 0. \quad (4.169)$$

Then the 2-point functions corresponding to external lines are those of Figure 4.5.

As in 3.2.3, any two diagrams G_1, G_2 are considered as **equal** if they differ only by their diagrammatical realization resp. are called **equivalent** ($G_1 \cong G_2$) if they differ only by the distribution of their vertex indices.

If all integrals over internal momenta exist⁷⁸ (absence of ultraviolet divergences) then \hat{A}_G can be evaluated by the following naive FEYNMAN **rules** of QED:

1. Assign suitable momenta to every line of G and then replace it by a corresponding factor:

$$-i(2\pi)^{-4} \epsilon(k-j) \lim_{\epsilon \rightarrow +0} \frac{(\gamma^\nu q_\nu + m)|_{r_k l_j}}{q^2 - m^2 + i\epsilon} \hat{=} k \bullet \xrightarrow{q} \xleftarrow{-q} \bullet j$$

$$-g^{\mu_j \mu_k} i(2\pi)^{-4} \lim_{\epsilon \rightarrow +0} \frac{1}{k^2 + i\epsilon} \hat{=} j \bullet \overset{\text{wavy}}{\text{---}} \underset{-k}{\bullet} k$$

$$(2\pi)^{-\frac{3}{2}} \epsilon'^{\mu_j}(\mathbf{k}') \hat{=} \begin{array}{c} \mathbf{k}' \quad \epsilon' \\ | \\ \bullet j \\ -\mathbf{k}' \end{array}$$

$$\sqrt{2m}(2\pi)^{-\frac{3}{2}} (\omega_\sigma^{(-)}(-\mathbf{p})^* \gamma^0)_{l_j} \hat{=} \begin{array}{c} \mathbf{p} \quad j \\ | \\ \bullet \\ \mathbf{p} \quad \sigma \end{array}$$

etc.

2. For $j = 1, \dots, V_G$ replace vertex j by the factor⁷⁹

$$-ie (\gamma_{\mu_j})_{l_j r_j} (2\pi)^4 \delta(P)$$

where

$$P \stackrel{\text{def}}{=} \left\{ \begin{array}{l} \text{sum of all 4-momenta}^{80} \text{ assigned to line ends} \\ \text{connected with vertex } j. \end{array} \right.$$

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⁷⁷Here \mathbf{e}_j and $\hat{a}^j(\mathbf{p})$ are to be understood in the sense of (4.57).

⁷⁸This is the case if G is a so-called **tree diagram**, i.e. if it does not contain closed loops.

⁷⁹The δ -function results from integration over x_1, \dots, x_{V_G} .

$$\begin{aligned}
\left. \begin{array}{l} \mathbf{k}' \\ \epsilon' \\ \bullet \\ j \end{array} \right\} &\hat{=} \langle \Omega | \hat{a}_{\epsilon'}(\mathbf{k}') \hat{A}^{\mu_j}(x_j) \Omega \rangle \\
&= (2\pi)^{-\frac{3}{2}} \epsilon'^{\mu_j}(\mathbf{p}') e^{+ik'x_j} \Big|_{k^0=|\mathbf{k}'|} \\
\left. \begin{array}{l} \bullet \\ j \\ \mathbf{k} \\ \epsilon \end{array} \right\} &\hat{=} \langle \Omega | \hat{A}^{\mu_j}(x_j) \hat{a}_{\epsilon}^{\dagger}(\mathbf{k}) \Omega \rangle \\
&= (2\pi)^{-\frac{3}{2}} \epsilon^{\mu_j}(\mathbf{p}) e^{-ikx_j} \Big|_{k^0=|\mathbf{k}|} \\
\left. \begin{array}{l} \mathbf{p}' \\ \sigma' \\ \downarrow \\ \bullet \\ j \end{array} \right\} &\hat{=} \langle \Omega | \hat{b}_{\sigma'}(\mathbf{p}') \left(\hat{\Psi}(x_j)^{\dagger} \gamma^0 \right)_l \Omega \rangle \\
&= \sqrt{2m} (2\pi)^{-\frac{3}{2}} \left(\omega_{\sigma'}^{(+)}(\mathbf{p}')^* \gamma^0 \right)_l e^{+ip'x_j} \Big|_{p^0=\omega_{\mathbf{p}'}} \\
\left. \begin{array}{l} \bullet \\ j \\ \mathbf{p} \\ \uparrow \\ \sigma \end{array} \right\} &\hat{=} \langle \Omega | \left(\hat{\Psi}(x_j) \right)_r \hat{b}_{\sigma}(\mathbf{p})^{\dagger} \Omega \rangle \\
&= \sqrt{2m} (2\pi)^{-\frac{3}{2}} \left(\omega_{\sigma}^{(+)}(\mathbf{p}) \right)_r e^{-ipx_j} \Big|_{p^0=\omega_{\mathbf{p}}} \\
\left. \begin{array}{l} \mathbf{p}' \\ \sigma' \\ \downarrow \\ \bullet \\ j \end{array} \right\} &\hat{=} \langle \Omega | \hat{d}_{\sigma'}(\mathbf{p}') \left(\hat{\Psi}(x_j) \right)_r \Omega \rangle \\
&= \sqrt{2m} (2\pi)^{-\frac{3}{2}} \left(\omega_{\sigma'}^{(-)}(-\mathbf{p}') \right)_r e^{+ip'x_j} \Big|_{p^0=\omega_{\mathbf{p}'}} \\
\left. \begin{array}{l} \bullet \\ j \\ \mathbf{p} \\ \uparrow \\ \sigma \end{array} \right\} &\hat{=} \langle \Omega | \left(\hat{\Psi}(x_j)^{\dagger} \gamma^0 \right)_l \hat{d}_{\sigma}^{\dagger}(\mathbf{p}) \Omega \rangle \\
&= \sqrt{2m} (2\pi)^{-\frac{3}{2}} \left(\omega_{\sigma}^{(-)}(-\mathbf{p})^* \gamma^0 \right)_l e^{-ipx_j} \Big|_{p^0=\omega_{\mathbf{p}}} \\
\left. \begin{array}{l} \mathbf{k}' \\ \epsilon' \\ \mathbf{k} \\ \epsilon \end{array} \right\} &\hat{=} \langle \Omega | \hat{a}_{\epsilon'}(\mathbf{k}') \hat{a}_{\epsilon}(\mathbf{k})^{\dagger} \Omega \rangle = 2|\mathbf{k}| \epsilon'(\mathbf{k}') \cdot \epsilon(\mathbf{k}) \delta(\mathbf{k} - \mathbf{k}') \\
\left. \begin{array}{l} \mathbf{p}' \\ \sigma' \\ \mathbf{p} \\ \uparrow \\ \sigma \end{array} \right\} &\hat{=} \langle \Omega | \hat{b}_{\sigma'}(\mathbf{p}') \hat{b}_{\sigma}(\mathbf{p})^{\dagger} \Omega \rangle = 2\omega_{\mathbf{p}} \delta_{\sigma\sigma'} \delta(\mathbf{p} - \mathbf{p}') \\
\left. \begin{array}{l} \mathbf{p}' \\ \sigma' \\ \mathbf{p} \\ \downarrow \\ \sigma \end{array} \right\} &\hat{=} \langle \Omega | \hat{d}_{\sigma'}(\mathbf{p}') \hat{d}_{\sigma}(\mathbf{p})^{\dagger} \Omega \rangle = 2\omega_{\mathbf{p}} \delta_{\sigma\sigma'} \delta(\mathbf{p} - \mathbf{p}')
\end{aligned}$$

Figure 4.5: External lines of QED

3. Take the product of all factors and sum over all indices r_j, l_j, μ_j and integrate over all momenta assigned to internal lines (only one integration per internal line).

4. Finally multiply by $\sigma_G \in \{+1, -1\}$ to be determined as follows:

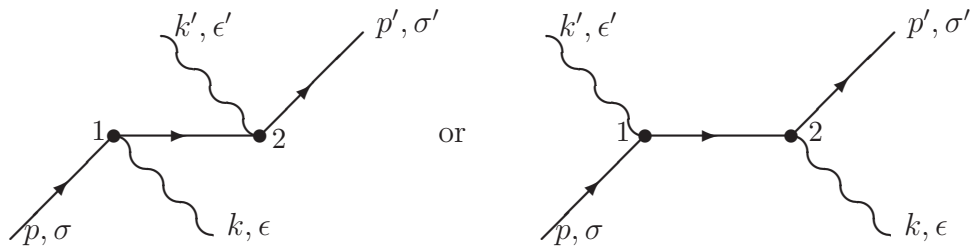
Write down the corresponding contraction scheme, e.g.

$$\hat{c}_{j_{N'}}(\mathbf{p}'_{N'}) \cdots \hat{c}_{j_1}(\mathbf{p}'_1) : \hat{S}_1(x_1) \cdots \hat{S}(x_{V_G}) : \hat{c}_{k_1}^\dagger(\mathbf{p}_1) \cdots \hat{c}_{k_N}^\dagger(\mathbf{p}_N),$$

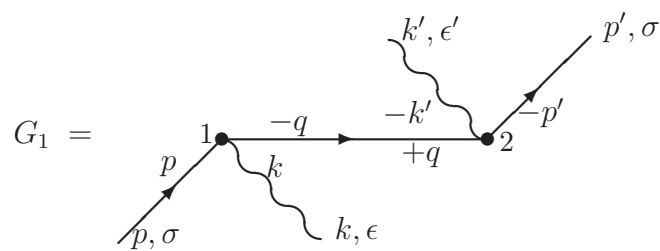
and rearrange the operators such that all contracted pairs become direct neighbors without changing the relative order of the operators forming any such pair. Then σ_G is the signum of the overall permutation resulting this way.

4.3.4 Example: COMPTON Scattering

For COMPTON *scattering*, i.e. for electron-photon scattering in second order⁸¹ of perturbation theory, only graphs equivalent to



are relevant. The factors corresponding to the graph



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⁸⁰Here, the 3-momenta assigned to external lines have to lifted to the mass shell of the asymptotic ‘particle’ type.

⁸¹This means that only contributions with $V_G \leq 2$ are considered.

are the following:

$$\begin{aligned}
& \sqrt{2m} (2\pi)^{-3/2} \left(\omega_{\sigma}^{(+)}(\mathbf{p}) \right)_{r_1} \\
& (2\pi)^{-3/2} \epsilon^{\mu_1}(\mathbf{k}), \\
& -ie (\gamma_{\mu_1})_{l_1 r_1} (2\pi)^4 \delta(p+k-q), \\
& -i (2\pi)^{-4} \lim_{\epsilon \rightarrow +0} \frac{(\gamma^{\nu} q_{\nu} + m)_{r_2 l_1}}{q^2 - m^2 + i\epsilon} \\
& -ie (\gamma_{\mu_2})_{l_2 r_2} (2\pi)^4 \delta(q-p'-k'), \\
& (2\pi)^{-3/2} \epsilon'^{\mu_2}(\mathbf{k}'), \\
& \sqrt{2m} (2\pi)^{-3/2} \left(\omega_{\sigma'}^{(+)}(\mathbf{p}')^* \gamma^0 \right)_{l_2}.
\end{aligned}$$

The signum of the permutation mapping

$$\hat{b}_{\sigma'}(\mathbf{p}') \hat{\Psi}(x_1)^{\dagger} \hat{\Psi}(x_1) \hat{\Psi}(x_2)^{\dagger} \hat{\Psi}(x_2) \hat{b}_{\sigma}^{\dagger}(\mathbf{p})$$

onto

$$\left(\hat{\Psi}(x_1) \hat{b}_{\sigma}^{\dagger}(\mathbf{p}) \right) \left(\hat{\Psi}(x_1)^{\dagger} \hat{\Psi}(x_2) \right) \left(\hat{b}_{\sigma'}(\mathbf{p}') \hat{\Psi}(x_2)^{\dagger} \right)$$

is $\sigma_{G_1} = -1$. Therefore

$$\begin{aligned}
\hat{A}_{G_1} &= \frac{me^2}{2i\pi^2} \lim_{\epsilon \rightarrow +0} \int \delta(p+k-q) \delta(q-p'-k') \omega_{\sigma'}^{(+)}(\mathbf{p}')^* \gamma^0 \not{\epsilon}'(\mathbf{k}') \cdot \\
& \quad \cdot \frac{\not{q} + m}{q^2 - m^2 + i\epsilon} \not{\epsilon}(\mathbf{k}) \omega_{\sigma}^{(+)}(\mathbf{p}) d\mathbf{q} \\
&= \frac{me^2}{2i\pi^2} \delta(p+k-p'-k') \omega_{\sigma'}^{(+)}(\mathbf{p}')^* \gamma^0 \not{\epsilon}'(\mathbf{k}') \frac{\not{p} + \not{k} + m}{2p \cdot k} \not{\epsilon}(\mathbf{k}) \omega_{\sigma}^{(+)}(\mathbf{p}),
\end{aligned}$$

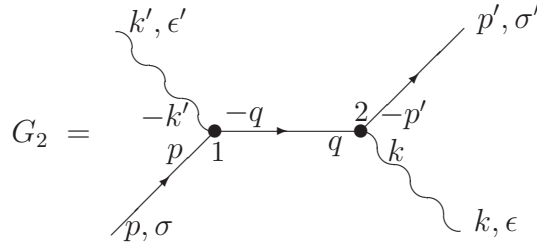
where we use the usual conventions

$$\not{x} \stackrel{\text{def}}{=} \gamma^{\mu} x_{\mu}, \quad \not{p} \stackrel{\text{def}}{=} \gamma^{\mu} p_{\mu} \quad \text{etc.}$$

Similarly we get

$$\hat{A}_{G_2} = \frac{me^2}{2i\pi^2} \delta(p+k-p'-k') \omega_{\sigma'}^{(+)}(\mathbf{p}')^* \gamma^0 \not{\epsilon}'(\mathbf{k}') \frac{\not{p}' - \not{k}' + m}{-2p \cdot k'} \not{\epsilon}'(\mathbf{k}') \omega_{\sigma}^{(+)}(\mathbf{p})$$

for



and hence

$$\begin{aligned}
 & \left\langle \hat{b}_{\sigma'}(\mathbf{p}') \hat{a}_{\epsilon'}(\mathbf{k}') \Omega \mid \left(\hat{S}_0 - \hat{1} \right)_{2. \text{ order}} \hat{b}_{\sigma}(\mathbf{p}) \dagger \hat{a}_{\epsilon}(\mathbf{k}) \dagger \Omega \right\rangle \\
 &= \frac{me^2}{2i\pi^2} \delta(p+k-p'-k') \omega_{\sigma'}^{(+)}(\mathbf{p}')^* \Gamma(p, k, k') \omega_{\sigma}^{(+)}(\mathbf{p}); \quad \text{where:} \\
 & \Gamma(p, k, k') \stackrel{\text{def}}{=} \gamma^0 \left(\not{\epsilon}'(\mathbf{k}') \frac{\not{p} + \not{k}' + m}{2p \cdot k} \not{\epsilon}(\mathbf{k}) + \not{\epsilon}(\mathbf{k}) \frac{\not{p} - \not{k}' + m}{-2p \cdot k'} \not{\epsilon}'(\mathbf{k}') \right).
 \end{aligned} \tag{4.170}$$

The cross section for unpolarized incoming and outgoing electrons⁸² is

$$\begin{aligned}
 & \sigma(p, k, \epsilon, \epsilon') \\
 &= \frac{(2\pi)^2}{8p \cdot k} \sum_{\sigma, \sigma' = \pm} \int \left| \frac{\left\langle \hat{b}_{\sigma'}(\mathbf{p}') \hat{a}_{\epsilon'}(\mathbf{k}') \Omega \mid \left(\hat{S}_0 - \hat{1} \right)_{2. \text{ order}} \hat{b}_{\sigma}(\mathbf{p}) \dagger \hat{a}_{\epsilon}(\mathbf{k}) \dagger \Omega \right\rangle}{\delta(p' + k' - p - k)} \right|^2 \times \\
 & \quad \times \delta(p' + k' - p - k) \frac{d\mathbf{k}'}{2k'^0} \frac{d\mathbf{p}'}{2p'^0},
 \end{aligned} \tag{4.171}$$

where p resp. k is the momentum of the incoming electron resp. photon and ϵ resp. ϵ' is the polarization of the incoming resp. outgoing photon. Note that all 4-momenta have to be on the corresponding mass shell:

$$k^0 = |\mathbf{k}|, \quad k'^0 = |\mathbf{k}'|, \quad p^0 = \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}, \quad p'^0 = \omega_{\mathbf{p}'}$$

In order to evaluate (4.171) we have to calculate

$$I \stackrel{\text{def}}{=} \int \sum_{\sigma, \sigma' = \pm} \left| \omega_{\sigma'}^{(+)}(\mathbf{p}')^* \Gamma(p, k, k') \omega_{\sigma}^{(+)}(\mathbf{p}) \right|^2 \delta(p' + k' - p - k) \frac{d\mathbf{p}'}{2\omega_{\mathbf{p}'}} \frac{d\mathbf{k}'}{2|\mathbf{k}'|}. \tag{4.172}$$

By $\mathbf{p} = 0$ and $\epsilon^0(\mathbf{k}) = \epsilon'^0(\mathbf{k}') = 0$, (4.108) gives

$$(\not{p} + m) \not{\epsilon}^{(\prime)}(\mathbf{p}) \omega_{\sigma}^{(+)}(\mathbf{p}) = - \not{\epsilon}^{(\prime)}(\mathbf{p}) (\not{p} - m) \omega_{\sigma}^{(+)}(\mathbf{p}) = 0.$$

With

$$\left(\omega'^* \hat{\Gamma} \omega \right)^{*T} = \left(\omega'^* \right)^* \left(\hat{\Gamma}^* \right)^T \omega^{*T} = \left(\omega^* \hat{\Gamma}^* \omega' \right)^T$$

this implies

$$\left| \omega'^* \hat{\Gamma} \omega \right|^2 = \left(\omega'^* \hat{\Gamma} \omega \right) \left(\omega^* \hat{\Gamma} \omega' \right).$$

In the standard representation (4.115) the latter together with

$$\sum_{\sigma = \pm} \left(\omega_{\sigma}^{(+)}(\mathbf{p}) \right)_r \left(\omega_{\sigma}^{(+)}(\mathbf{p})^* \gamma^0 \right)_l = \left(\frac{\not{p} + m}{2m} \right)_{rl} \tag{4.173}$$

⁸²This means summation of the polarizations of the outgoing electron and averaging over the polarizations of the incoming electron. Therefore, we need an additional factor $\frac{1}{2}$ compared to the formula of Exercise 53.

implies

$$I = \int \frac{d\mathbf{p}'}{2\omega_{\mathbf{p}'}} \frac{d\mathbf{k}'}{2|\mathbf{k}'|} \delta(p' + k' - p - k) \text{Tr} \left(\gamma^0 \hat{\Gamma}(p, k, k')^* \frac{\not{p}' + m}{2m} \right),$$

where

$$\hat{\Gamma}(p, k, k') \stackrel{\text{def}}{=} \gamma^0 \left(\frac{\not{\epsilon}'(\mathbf{k}') \not{k}' \not{\epsilon}(\mathbf{k})}{2|\mathbf{k}|m} + \frac{\not{\epsilon}(\mathbf{k}) \not{k}' \not{\epsilon}'(\mathbf{k}')}{2|\mathbf{k}'|m} \right) \quad (4.174)$$

and hence

$$\hat{\Gamma}(p, k, k')^* = \gamma^0 \left(\frac{\not{\epsilon}(\mathbf{k}) \not{k}' \not{\epsilon}'(\mathbf{k}')}{2|\mathbf{k}|m} + \frac{\not{\epsilon}'(\mathbf{k}') \not{k}' \not{\epsilon}(\mathbf{k})}{2|\mathbf{k}'|m} \right) \quad (4.175)$$

(because of $\boldsymbol{\gamma} = -\boldsymbol{\gamma}^*$ and $\gamma^0 = \gamma^{0*}$).

In the laboratory system, i.e. for

$$\mathbf{p} = 0, \quad p^0 = m,$$

the conditions

$$p' + k' = p + k, \quad p'^2 = m^2, \quad k'^2 = 0$$

are well known to imply the so-called **COMPTON condition**

$$\boxed{k'^0 = |\mathbf{k}'| = \frac{|\mathbf{k}|}{1 + \frac{|\mathbf{k}|}{m}(1 - \cos \vartheta)}, \quad \cos \vartheta \stackrel{\text{def}}{=} \frac{\mathbf{k}}{|\mathbf{k}|} \cdot \frac{\mathbf{k}'}{|\mathbf{k}'|}. \quad (4.176)}$$

Therefore p' and k' are uniquely fixed, in the laboratory system, by \mathbf{k} and the direction of \mathbf{k}' . Hence, there is a function $f(|\mathbf{k}|, (\vartheta, \varphi))$ with

$$f(|\mathbf{k}|, (\vartheta, \varphi)) = \text{Tr} \left(\gamma^0 \hat{\gamma}(p, k, k')^* \frac{\not{p}' + m}{2m} \right). \quad (4.177)$$

(ϑ, φ) are polar angles of \mathbf{k}' (with $\vartheta = 0$ for $\mathbf{k} \parallel \mathbf{k}'$). For $\mathbf{p} = 0$, therefore,

$$\begin{aligned} I &= \int \frac{|\mathbf{k}'|^2 d|\mathbf{k}'| d\omega}{2|\mathbf{k}'|} \int dp' \theta(p'^0) \delta(p'^2 - m^2) \delta(p' + k' - p - k) f(k, \vartheta, \varphi) \\ &= d\omega d|\mathbf{k}'| \frac{|\mathbf{k}'|}{2} \theta(p^0 + k^0 - k'^0) \delta((p + k - k')^2 - m^2) f(k, \vartheta, \varphi) \\ &= \int d\omega \int_0^{p^0 + k^0} d|\mathbf{k}'| \frac{|\mathbf{k}'|}{2} \delta(2m|\mathbf{k}| - |\mathbf{k}'|(2m + 2|\mathbf{k}|(1 - \cos \vartheta))) f(k, \vartheta, \varphi) \\ &\quad \text{(upper boundary redundant due to (4.176))} \\ &= \int d\omega \frac{m|\mathbf{k}|}{4(m + |\mathbf{k}|(1 - \cos \vartheta))^2} f(k, \vartheta, \varphi). \end{aligned}$$

By (4.176), this implies

$$\frac{dI}{d\Omega} = \frac{|\mathbf{k}'|^2}{4m|\mathbf{k}|} f(k, \vartheta, \varphi). \quad (4.178)$$

(4.177) and (4.172)/(4.175), on the other hand, imply

$$f(k, \vartheta, \varphi) = \text{Tr} \left(\frac{\not{p}' + m}{2m} \left(\frac{\not{\epsilon}'(k') \not{k} \not{\epsilon}(k)}{2|\mathbf{k}|m} + \frac{\not{\epsilon}(k) \not{k}' \not{\epsilon}'(k')}{2|\mathbf{k}'|m} \right) \times \right. \\ \left. \times \frac{\not{p} + m}{2m} \left(\frac{\not{\epsilon}(k) \not{k} \not{\epsilon}'(k')}{2|\mathbf{k}|m} + \frac{\not{\epsilon}'(k') \not{k}' \not{\epsilon}(k)}{2|\mathbf{k}'|m} \right) \right). \quad (4.179)$$

Calculation of this trace is easily done by computer algebra. In Section 17.7 of the REDUCE⁸³ manual there is already a listing of the corresponding program:

```
ON DIV;
MASS K= 0, KP= 0, P= MC, PP= MC; VECTOR EP,E;
MSHELL K,KP,P,PP;
LET P.EP= 0, P.E= 0, P.PP= MC**2+K.KP, P.K= MC*NK,P.KP=
    MC*NKP, PP.EP= -KP.EP, PP.E= K.E, PP.K= MC*NKP, PP.KP=
    MC*NK, K.EP= 0, K.KP= MC*(NK-NKP), KP.E= 0, EP.EP= -1, E.E=-1;
(G(L,PP) + MC)/(2*MC)*(G(L,E,EP,K)/(2*K.P) + G(L,EP,E,KP)/(2*KP.P))
* (G(L,P) + MC)/(2*MC)*(G(L,K,EP,E)/(2*K.P) + G(L,KP,E,EP)/(2*KP.P))$
WRITE "1/4 Trace = ",WS;
```

This gives the following result for (4.179):

$$1/4 \text{ Trace} = \text{MC}^{(-2)} * (1/2*EP.E^2 + 1/8*NKP*NK^{(-1)} + 1/8*NKP^{(-1)}*NK - 1/4)$$

The explicit meaning of this REDUCE message is:

$$\frac{1}{4} f(k, \vartheta, \varphi) = (mc)^{-2} \left(\frac{1}{2} (\epsilon' \cdot \epsilon)^2 + \frac{1}{8} \frac{|\mathbf{k}'|}{|\mathbf{k}|} + \frac{1}{8} \frac{|\mathbf{k}|}{|\mathbf{k}'|} - \frac{1}{4} \right).$$

Since we use natural units this, together with (4.178), proves the so-called **KLEIN-NISHINA formula**:

$$\frac{d}{d\Omega} \sigma(p, k, \epsilon, \epsilon')|_{p=0} = \frac{1}{4} (r_0)^2 \left(\frac{|\mathbf{k}'|}{|\mathbf{k}|} \right)^2 \left(\frac{|\mathbf{k}'|}{|\mathbf{k}|} + \frac{|\mathbf{k}|}{|\mathbf{k}'|} - 2 + 4 (\epsilon'(\mathbf{k}') \cdot \epsilon(\mathbf{k}))^2 \right),$$

where: $r_0 \stackrel{\text{def}}{=} \text{classical radius of the electron}$

$$= \frac{e^2}{4\pi m} \quad \text{in natural units in the HEAVISIDE system}$$

$$\cong 2,82 \dots \cdot 10^{-13} \text{cm}.$$

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⁸³For MATHEMATICA the package TRACER by M. JAMIN and M.E. LAUTENBACHER (Jamin and Lautenbacher, 1993) is useful in this context.

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